arXiv:0804.4728v1 [math.RA] 30 Apr 2008

# REPRESENTATIONS OF $Aut(A(\Gamma))$ ACTING ON HOMOGENEOUS COMPONENTS OF $A(\Gamma)$ AND $A(\Gamma)^!$

COLLEEN DUFFY

ABSTRACT. In this paper we will study the structure of algebras $A(\Gamma)$ associated to two directed, layered graphs $\Gamma$. These are algebras associated with Hasse graphs of n-gons and the algebras $Q_n$ related to pseudoroots of noncommutative polynomials. We will find the filtration preserving automorphism group of these algebras and then we will find the multiplicities of the irreducible representations of $Aut(A(\Gamma))$ acting on the homogeneous components of $A(\Gamma)$ and $A(\Gamma)^!$.

## Introduction

In this paper we will be considering directed graphs $\Gamma$ satisfying certain hypotheses. There exists an algebra $A(\Gamma)$ over a field $k$, an associated graded algebra $\text{gr}A(\Gamma)$, and a dual algebra $A(\Gamma)^!$ of $\text{gr}A(\Gamma)$ associated with each of these graphs. In Section 1 we will give some preliminaries on how these algebras are built from the graphs as well as a basis for $A(\Gamma)$. The definition of the dual algebra and of subalgebras that will play an integral role in what follows will be given in Section 2. In Section 3 we will introduce the two algebras $A(\Gamma_{D_n})$ and $Q_n$ that we will be describing throughout this paper.

The automorphism group of the graph injects into the automorphism group $\text{Aut}A(\Gamma)$ of $A(\Gamma)$. Furthermore, the nonzero scalars inject into the automorphism group of the algebra. Thus, one is naturally led to ask how these automorphism groups are related. This question will be answered in Section 4.

A second question that we are led to is to describe the homogeneous components of $A(\Gamma)$ and $A(\Gamma)^!$. We will decompose the graded algebra and its dual into irreducible $\text{Aut}(A(\Gamma))$-modules by calculating the graded trace generating functions. These graded trace generating functions are actually the graded dimensions of certain subalgebras of $\text{gr}A(\Gamma)$. Hence, the technique for calculating the graded trace generating functions is to abstract the problem into finding the graded dimension of subalgebras. In Section 5 we will explain why this is true and how these graded dimensions are found.

The graded traces of our two particular algebras will be given in Section 6 and their decompositions in Section 7. The decomposition can be found using the graded trace and the characters of the automorphism group of the algebra. The dual algebras of these two examples will be considered in Section 8.

*Date*: April 2008.
I would like to thank my advisor, Prof. Robert Wilson.

# 1. Preliminaries

## 1.1. The Algebra $A(\Gamma)$

Certain algebras, denoted $A(\Gamma)$, associated to directed graphs were first defined by Gelfand, Retakh, Serconek, and Wilson [GRSW]. We recall the definitions of $A(\Gamma)$ and $\mathrm{gr}A(\Gamma)$ following the development found in [[RSW], §2]. Let k be a field and for any set $W$ let $T(W)$ be the free associative algebra on $W$ over k. Let $\Gamma = (V, E)$ be a directed graph where $V$ is the set of vertices, $E$ the set of edges, and there are functions $t, h : E \to V$ (tail and head of $e$). We say $\Gamma$ is a layered graph if $V = \bigcup_{i=0}^{n} V_i$, $E = \bigcup_{i=0}^{n} E_i$, $t : E_i \to V_i$, and $h : E_i \to V_{i-1}$. If $v \in V_i$ ($e \in E_i$), we say the level of $v$, (respectively $e$) is $i$; denote this by $|v|$, (resp. $|e|$). We will assume throughout this paper that $V_0 = \{*\}$ and that for all $v \in V_+ = \bigcup_{i=1}^{n} V_i$, there exists at least one $e \in E$ such that $t(e) = v$. For each $v \in V_+$, fix some $e_v \in E$ with $t(e_v) = v$; call this a distinguished edge.

A path from $v \in V$ to $w \in V$ is a sequence of edges $\pi = \{e_1, ..., e_m\}$ such that $t(e_1) = v$, $h(e_m) = w$, and $t(e_{i+1}) = h(e_i), 1 \leq i < m$. We will say $t(\pi) = v$, $h(\pi) = w$, and the length of $\pi, l(\pi)$, is $m$. Write $v > w$ if there exists a path from $v$ to $w$. For $\pi = \{e_1, ..., e_m\}$, define $e(\pi, k) := \sum_{1 \leq i_1 < \cdots < i_k \leq m} e_{i_1} \cdots e_{i_k}$. For each $v \in V$ there is a path $\pi_v = \{e_1, ..., e_m\}$, called the distinguished path, from $v$ to * defined by $e_1 = e_v$, $e_{i+1} = e_{h(e_i)}$ for $1 \leq i < m$, and $h(e_m) = *$. When $\pi_v$ is the distinguished path from $v$ to *, we will write $e(v, k)$ in lieu of $e(\pi_v, k)$. Let $R$ be the two-sided ideal of $T(E)$ generated by $\{e(\pi_1, k) - e(\pi_2, k) : t(\pi_1) = t(\pi_2),\ h(\pi_1) = h(\pi_2),\ 1 \leq k \leq l(\pi_1)\}$.

**Definition.** $A(\Gamma) = T(E)/R$

Let $\hat{e}(v, k)$ denote the image in $A(\Gamma)$ of $e_1 \cdots e_k$. Finally say that $(v, k)$ covers $(w, l)$ if $v > w$ and $k = |v| - |w|$, write this as $(v, k) \gtrdot (w, l)$.

**Theorem 1.1.** *[[RSW], Thm 1] - Let $\Gamma = (V, E)$ be a layered graph, $V = \bigcup_{i=0}^{n} V_i$, $V_0 = \{*\}$. Then $\mathcal{B}(\Gamma) := \{\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) : l \geq 0, v_1, ..., v_l \in V_+, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not\gtrdot (v_{i+1}, k_{i+1})\}$ is a basis for $A(\Gamma)$.*

There is also a presentation of $A(\Gamma)$ as a quotient of $T(V_+)$ [[RSW2],§3]. Every edge may be expressed as a linear combination of distinguished edges, and the distinguished edge $e_v$ may be identified with $v \in V_+$. Define $S_1(v) := \{w \in V_{|v|-1} : v > w\}$. A layered graph is uniform if for every $v \in V_j$, $j \geq 2$, every pair of vertices $u, w$ in $S_1(v)$ satisfies $S_1(u) \cap S_1(w) \neq \emptyset$ ("diamond condition").

**Proposition 1.2.** *[[RSW2],Prop 3.5] Let $\Gamma$ be a uniform layered graph. Then $A(\Gamma) \cong T(V_+)/R_V$ where $R_V$ is the two-sided ideal generated by*

$$\{v(u - w) - u^2 + w^2 + (u - w)x : v \in \bigcup_{i=2}^{n} V_i, u, w \in S_1(v), x \in S_1(u) \cap S_1(w)\}.$$

Remark: From now on we will just write $e(v, k)$ for $\hat{e}(v, k)$.

### 1.2. **Associated Graded Algebra** $grA(\Gamma)$

Next we will describe a filtration and grading on $A(\Gamma)$. Here we will also denote by $V$ the span of $V$ in $T(V)$, and by $E$ the span of $E$ in $T(E)$, when no confusion will arise. Let $W = \sum_{k\geq 0} W_k$ be a graded vector space (in our case $W = V$ or $E$). Then $T(W)$ is bigraded. One grading $T(W) = \sum_i T(W)_{[i]}$ is given by degree in the tensor algebra; i.e., $T(W)_{[i]} = span\{w_1 \cdots w_i : w_1, ..., w_i \in W\}$. The other grading is given by $T(W) = \sum_{i\geq 0} T(W)_i$ where
$T(W)_i = \text{span}\{w_1 \cdots w_r : r \geq 0, w_j \in W_{l_j}, l_1 + ... + l_r = i\}$.
The second grading induces an increasing filtration on $T(W)$:
$T(W)_{(i)} = \text{span}\{w_1 \cdots w_r : r \geq 0, w_j \in W_{l_j}, l_1 + \cdots + l_r \leq i\} = T(W)_0 + \cdots + T(W)_i$.

Because $T(W)_{(i)}/T(W)_{(i-1)} \cong T(W)_i$, $T(W)$ can be identified with its associated graded algebra. Define a map $gr : T(W)\backslash\{0\} \to T(W)\backslash\{0\} = grT(W)$ by $w = \sum_{i=0}^{k} w_i \mapsto w_k$ where $w_i \in T(W)_i$, $w_k \neq 0$. Of course, gr is not an additive map. [[RSW2],§2]

**Lemma 1.3.** *[[RSW2],Lemma 2.1] Let $W$ be a graded vector space and $I$ an ideal in $T(W)$. Then $gr(T(W)/(I)) \cong T(W)/(grI)$.*

Thus the associated graded algebra of $A(\Gamma)$, $grA(\Gamma)$, is isomorphic to $T(E)/grR$. The graded relations, $grR$, are for $\pi_1 = \{e_1, ..., e_m\}$ and $\pi_2 = \{f_1, ..., f_m\}$, $\{e_1 \cdots e_k = f_1 \cdots f_k,\ 1 \leq k \leq m\}$ (the leading term of $e(v,k)$). Another way to consider this is that $e(v, k+l) = e(v,k)e(u,l)$ in $grR$ where $v > u$. Recalling the definition of $\mathcal{B}(\Gamma)$ from Theorem 1.1, we see that $\{gr(b) : b \in \mathcal{B}(\Gamma)\}$ is a basis for $grA(\Gamma)$.

Let us now look at our second description of $A(\Gamma)$ as isomorphic to $T(V_+)/R_V$.

**Proposition 1.4.** *[[RSW2],Prop 3.6] Let $\Gamma$ be a uniform layered graph.
Then $grA(\Gamma) \cong T(V_+)/grR_V$ where $grR_V$ is generated by $\{v(u-w) : v \in \bigcup_{i=2}^{n} V_i, u, w \in S_1(v)\}$.*

Also, $A(\Gamma)_i = (T(E)_i + R)/R = (T(V_+)_i + R_V)/R_V$ and $A(\Gamma)_{(i)} = (T(E)_{(i)} + R)/R = (T(V_+)_{(i)} + R_V)/R_V$.

## 2. The Dual $A(\Gamma)^!$ and the Subalgebra $A(\Gamma^\sigma)$

### 2.1. **The Dual** $A(\Gamma)^!$

**Definition** $(A^!)$**.** *[[BVW],§ 2] Let $A = T(E)/(R)$, $R \subseteq E^{\otimes 2}$. Then $A^! = T(E^*)/(R^\perp)$ where $E^*$ is the dual vector space of $E$ and $R^\perp$ is the annihilator of $R$; i.e. $R^\perp = \{f \in (E^{\otimes 2})^* : f(x) = 0\, \forall x \in R\}$ of $(E^{\otimes 2})^*$ where $(E^{\otimes 2})^*$ is canonically identified with $E^{*\otimes 2}$.*

**Definition** $(A(\Gamma)^!)$**.** [D] *The dual of $grA(\Gamma)$ is $A(\Gamma)^! := T(E^*)/(grR)^\perp$.*

The dual element to the generator $e(v,k)$ in $A(\Gamma)$ will be denoted $e(v,k)^*$.

**Proposition 2.1.** [D]$A(\Gamma)^!$ *has a presentation with generators* $\{e(v,1)^*\}$ *and relations* $\{e(v,1)^* e(u,1)^* : v \not> u\} \cup \{e(v,1)^* \sum_{v > u} e(u,1)^*\}$

## 2.2. **The Subalgebra** $A(\Gamma^\sigma)$

We will now define a subalgebra of $grA(\Gamma)$. Let $\sigma$ be an automorphism of the layered graph $\Gamma$; i.e. an automorphism that preserves each level of the graph. Define $\Gamma^\sigma := (V_\sigma, E_\sigma)$ where $V_\sigma$ is the set of vertices $v \in V$ such that $\sigma(v) = v$ and $E_\sigma$ is the set of edges that connect the vertices minimally. Here minimally means that there is an edge $e \in E_\sigma$ from $v$ to $w$, $v$, $w \in V_\sigma$ if and only if $v \geq u \geq w$, $u \in V_\sigma$, implies $u = v$ or $u = w$.

**Definition** $(A(\Gamma^\sigma))$**.** *Define* $A(\Gamma^\sigma)$ *to be span*$\{e(v_1, k_1) \cdots e(v_l, k_l) : l \geq 0, v_1, ..., v_l \in V_\sigma \backslash *, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not> (v_{i+1}, k_{i+1})\}$.

This set is, in fact, a basis. The elements are linearly independent because the set is a subset of a basis.

**Theorem 2.2.** $A(\Gamma^\sigma)$ *is a subalgebra of* $grA(\Gamma)$.
*A presentation for* $A(\Gamma^\sigma)$ *is given by generators* $G' = \{e'(v,k) : v \in V_\sigma, 1 \leq k \leq |v|\}$ *and relations* $R' = \{e'(v, k+l) - e'(v,k)e'(u,l) : v > u \in V_\sigma\}$.

*Proof.* Define $\phi : T(G') \to grA(\Gamma)$ by $\phi(e'(v,k)) = e(v,k)$. We have $\phi(T(G')) \supseteq A(\Gamma^\sigma)$ because elements of $B_\sigma$ are formed from products of elements in $G'$.

In $A(\Gamma)$ we have

$$(*) \quad e(v,k)e(u,l) - e(v,k+l) \equiv \sum_{\substack{i_0, i_{r+1} \geq 0, i_1, ..., i_r \geq 1 \\ i_0 < k, i_0 + \cdots + i_r \leq k \\ i_0 + \cdots + i_{r+1} = k+l}} (-1)^{r+1} e(v, i_0) e(u, i_1) \cdots e(u, i_{r+1})$$

mod R [[GRSW], p6]. However, the elements on the right-hand side are all of lower degree than those on the left-hand side [[GRSW], Lemma 2.2]. Note that the elements on the left-hand side have degree k+l and are in $(T(E)/R)_{[(k+l)|v| - (k+l)(k+l+1)/2]}$. Therefore, in $grA(\Gamma)$, the terms on the right-hand side are zero. Hence, we have $e(v,k)e(u,l) - e(v,k+l) \equiv 0$. Consequently $\phi(e'(v,k)e'(u,l)) = \phi(e'(v,k+l))$.

Let $b' = e'(v_1,k_1) \cdots e'(v_l,k_l)$ be a monomial in $T(G')$. In $T(G')/ < R' >$, we may replace every occurrence of $e'(v_i,k_i)e'(v_{i+1},k_{i+1})$ such that $(v_i,k_i) > (v_{i+1},k_{i+1})$ in $b'$ with $e'(v_i, k_i + k_{i+1})$. Thus $b' \equiv e'(v'_1,k'_1) \cdots e'(v'_l,k'_l)$ such that $(v'_i,k'_i) \not> (v'_{i+1},k'_{i+1})$ in $T(G')/ < R' >$. Hence $\phi(b') \in A(\Gamma^\sigma)$, and so $\phi(T(G')) = A(\Gamma^\sigma)$.

By (*), $R' \subseteq \ker\phi$ and we have an induced surjective homomorphism $\phi' : T(G')/ < R' > \to A(\Gamma^\sigma)$.

Let $f = \sum k_i b'_i \in \ker\phi'$, where $k_i$ is an element in the field and $b'_i$ a monomial in $T(G')/ < R' >$. Then $0 = \phi'(f) = \sum k_i \phi'(b'_i) = \sum k_i b_i$ is a linear combination of basis elements in $A(\Gamma^\sigma)$. This implies that $k_i = 0 \forall i$ and so $f = 0$. Therefore, $\phi'$ is an isomorphism. □

We will write $e(v,k)$ for $e'(v,k)$ from now on.

For $x$ a basis element of $A(\Gamma)_{[i]}$, write $\sigma(x)$ as a linear combination of basis elements and say the coefficient of $x$ in $\sigma(x)$ is $\alpha$. Denote this value $\alpha$ by $t_\sigma(x)$. Then, for finite-dimensional $A(\Gamma)_{[i]}$, $Tr_\sigma(A(\Gamma)_{[i]}) = \sum_{x \in \text{basis}} t_\sigma(x)$. In this paper we will be looking at the trace of $\sigma$ acting on $A(\Gamma)_{[i]}$ and $A(\Gamma)^!_{[i]}$.

# 3. Definition and Hilbert Series of Two Algebras

## 3.1. **Hasse graph of an n-gon:** $\Gamma_{D_n}$

A Hasse graph, or Hasse diagram, is a graph which represents a finite poset $\mathcal{P}$. The vertices in the graph are elements of $\mathcal{P}$ and there is an edge between $x$, $y \in \mathcal{P}$ if $x < y$ and there does not exist a $z \in \mathcal{P}$ such that $x < z < y$. Furthermore, the vertex for $x$, $v_x$, is in a lower level than that for $y$, $v_y$ (if we talk about layers in the graph, $|v_x| = |v_y| - 1$).

Consider a polytope. We can put a partial order on the set of k-faces in the polytope by $x < y$ if $x$ is an $(n-1)$-face, $y$ is an $n$-face and $x$ is a face of $y$.

Thus, the Hasse graph of an n-gon has one vertex in levels 0 and 3 and n vertices on levels 1 and 2. The top vertex is connected to all vertices in level 2 (all edges are in the 2-dimensional polygon), each vertex in level 2 is connected to the vertex directly below it and the one to that vertex's right, with wrapping around to the first vertex in level one for the last vertex in level two (each edge connects two adjacent vertices), and each vertex in level 1 is connected to the minimal vertex. Label the vertices by using subscripts in $\mathbb{Z}/(n)$. In level 1 call the vertices $w_1, ..., w_n$, call the vertices in level 2 $v_{12}, ..., v_{n1}$ (where the subscripts indicate to which vertices in level 1 the vertex is connected), and the top vertex is $u$. See Figure 1.

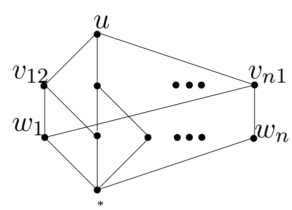


FIGURE 1. $\Gamma_{D_n}$

We consider the algebra $A(\Gamma_{D_n})$ determined by this graph. The construction of this algebra is described in [GRSW] (see § 1). In brief (using the definition given in Proposition 1.2), the generators are the vertices and the relations are that two paths which have the same starting and ending vertices are equivalent. These relations are, for $1 \leq i \leq n$,

1) $v_{ii+1}(w_i - w_{i+1}) - w_i^2 + w_{i+1}^2$

2) $u(v_{ii+1} - v_{i+1i+2}) - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1}$.

We will give here two bases for $A(\Gamma_{D_n})$, one for each of the two definitions of $A(\Gamma)$ given in Section 1. First we will give a basis in terms of the vertices (Proposition 1.2).

**Proposition 3.1.** *A basis $\mathcal{B}$ of $A(\Gamma_{D_n})$ consists of * and the set of all words in $u$, $v_{i\,i+1}$, and $w_i$ such that the following conditions on the words hold: the subword $v_{ii+1}w_j$ only occurs if $j \neq i+1$, the subword $uv_{ii+1}$ only if $i = 1$, and $uv_{ii+1}w_j$ only if $i = j = 1$.*

*Proof.* We can describe a basis of monomials for $A(\Gamma_{D_n})$ using Bergman's Diamond Lemma [Berg]. Put a partial order on the generators such that $u > v_{i\,i+1} > w_j \forall i,j$ and $v_{i\,i+1} > v_{j\,j+1}$ and $w_i > w_j$ if $i > j$. Order monomials lexicographically. The reductions are $uv_{i+1\,i+2} \equiv uv_{i\,i+1} - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1}$, $1 \le i \le n-1$ and $v_{ii+1}w_{i+1} \equiv v_{ii+1}w_i - w_i^2 + w_{i+1}^2$, $1 \le i \le n$.

We need to find a complete list of reductions so that all ambiguities resolve. The only ambiguity will occur when we have a word that ends in $v$ overlapping with one beginning with $v$; i.e. $uv_{i+1\,i+2}w_{i+2}$.

$(uv_{i+1\,i+2})w_{i+2} \equiv (uv_{ii+1} - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1})w_{i+2} \equiv uv_{ii+1}w_{i+2} - v_{ii+1}^2w_{i+2} + v_{i+1i+2}(v_{i+1i+2}w_{i+1} - w_{i+1}^2 + w_{i+2}^2) + (v_{ii+1}w_i - w_i^2 + w_{i+1}^2)w_{i+2} - v_{i+1i+2}w_{i+1}w_{i+2} \equiv uv_{ii+1}w_{i+2} - v_{ii+1}^2w_{i+2} + v_{i+1i+2}^2w_{i+1} - v_{i+1i+2}w_{i+1}^2 + v_{i+1i+2}w_{i+1}w_{i+2} - w_{i+1}^2w_{i+2} + w_{i+2}^3 + v_{ii+1}w_iw_{i+2} - w_i^2w_{i+2} + w_{i+1}^2w_{i+2} - v_{i+1i+2}w_{i+1}w_{i+2} = uv_{ii+1}w_{i+2} + v_{i+1i+2}^2w_{i+1} - v_{ii+1}^2w_{i+2} - v_{i+1i+2}w_{i+1}^2 + v_{ii+1}w_iw_{i+2} + w_{i+2}^3 - w_i^2w_{i+2}$

and

$u(v_{i+1\,i+2}w_{i+2}) \equiv u(v_{i+1i+2}w_{i+1} - w_{i+1}^2 + w_{i+2}^2) \equiv (uv_{ii+1} - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1})w_{i+1} - uw_{i+1}^2 + uw_{i+2}^2 \equiv uv_{ii+1}w_i - uw_i^2 + uw_{i+1}^2 - v_{ii+1}(v_{ii+1}w_i - w_i^2 + w_{i+1}^2) + v_{i+1i+2}^2w_{i+1} + v_{ii+1}w_iw_{i+1} - w_i^2w_{i+1} + w_{i+1}^3 - v_{i+1i+2}w_{i+1}^2 - uw_{i+1}^2 + uw_{i+2}^2 \equiv uv_{ii+1}w_i + uw_{i+2}^2 - uw_i^2 - v_{ii+1}^2w_i + v_{ii+1}w_i^2 - v_{ii+1}w_iw_{i+1} + w_i^2w_{i+1} - w_{i+1}^3 + v_{i+1i+2}^2w_{i+1} + v_{ii+1}w_iw_{i+1} - w_i^2w_{i+1} + w_{i+1}^3 - v_{i+1i+2}w_{i+1}^2 = uv_{ii+1}w_i + uw_{i+2}^2 - uw_i^2 + v_{i+1i+2}^2w_{i+1} - v_{ii+1}^2w_i - v_{i+1i+2}w_{i+1}^2 + v_{ii+1}w_i^2$.

Thus we need to add an additional reduction; namely, $uv_{ii+1}w_{i+2} \equiv uv_{ii+1}w_i + uw_{i+2}^2 - uw_i^2 + v_{ii+1}^2w_{i+2} - v_{ii+1}^2w_i - v_{ii+1}w_iw_{i+2} + v_{ii+1}w_i^2 - w_{i+2}^3 + w_i^2w_{i+2}$. This does not create additional ambiguities since this reduction ends in $w$ and we have no reductions which begin in $w$. Also, no reductions end in u. Thus, all ambiguities now resolve.

Therefore, by Bergman's Diamond Lemma, $A(\Gamma_{D_n})$ may be identified with the k-module of monomials which are irreducible under these reductions. Hence, $\mathcal{B}$ is a basis for $A(\Gamma_{D_n})$. □

Next follows a basis in terms of edges (Thm 1.1).

**Proposition 3.2.** $\mathcal{B}' = \{e(x_1,k_1)\cdots e(x_l,k_l) : l \ge 0, x_1,...,x_l \in \{u, v_{12},...,v_{n1}, w_1,...,w_n\},$
$1 \le k_i \le |x_i|, (x_i,k_i) \not\geqslant (x_{i+1},k_{i+1})\}$ *is a basis for* $A(\Gamma_{D_n})$.

*Proof.* This follows directly from Theorem 1.1. □

In the preliminaries we stated that the algebra is generated by distinguished edges and so we can identify the distinguished edges with the vertices which are their tails - $e_v$ is identified with $v$. Thus $e(v,k)$ can be expressed as a product of k vertices (recall we are writing $e(v,k)$ in lieu of $\hat{e}(v,k)$), and so there is a correlation between the bases $\mathcal{B}$, $\mathcal{B}'$ as follows:

$e(u,3) \leftrightarrow uv_{12}w_1 \qquad e(u,2) \leftrightarrow uv_{12} \qquad e(u,1) \leftrightarrow u$

$e(v_{ii+1},2) \leftrightarrow v_{ii+1}w_i \qquad e(v_{ii+1},1) \leftrightarrow v_{ii+1} \qquad e(w_i,1) \leftrightarrow w_i$

It is important to observe that in the associated graded algebra, $\sigma \in Aut(A(\Gamma))$ permutes the elements of gr$\mathcal{B}$ and gr$\mathcal{B}'$.

Recall that the Hilbert series gives the graded dimension of an algebra; the coefficient of $t^k$ is the k-th graded dimension (see Section 1 for the grading on our algebras). We write this as $H(t) = \sum dim(A(\Gamma)_{[k]})t^k$.

**Proposition 3.3.** *The Hilbert series for $A(\Gamma_{D_n})$ is*

$$H(t) = \frac{1}{1-(2n+1)t+(2n-1)t^2-t^3} = \frac{1-t}{1-(2n+2)t+4nt^2-2nt^3+t^4}$$

*Proof.* We will give two proofs of this proposition. The first one uses Proposition 3.1 and induction to count basis elements. This will give us a recursion that can then be written as a generating function. The second method of proof is much shorter and uses a theorem from [RSW] that gives a formula for the Hilbert series of an algebra associated to a directed, layered graph.

**Method 1:** By Proposition 3.1, there are $n(n-1)$ subwords of the form $v_{ii+1}w_j$ which can occur in an element of $\mathcal{B}$ and exactly one of the forms $uv_{i\,i+1}$ and $uv_{i\,i+1}w_j$. This means that there are $n$ subwords of the form $v_{ii+1}w_j$ which cannot occur, $n-1$ of the form $uv_{i\,i+1}$, and $n^2-1$ of the form $uv_{i\,i+1}w_j$.

Let $d_k = \dim(A(\Gamma_{D_n})_{[k]})$.

We will proceed by induction.

- $d_0 = 1$
- $d_1 = 2n+1$: Every word of length one belongs to the basis since all reducible subwords are of length greater than one. A basis is: $\{u,\, v_{ii+1},\, w_i : 1 \le i \le n\}$
- $d_2 = 4n^2+2n+2$: There are $(2n+1)^2$ elements of length two and $2n-1$ of them are reducible. Hence, the dimension is $(2n+1)^2-(2n-1) = 4n^2+2n+2$. A basis is: $\{w_iu,\, w_iv_{j\,j+1},\, w_iw_j,\, uu,\, uv_{1\,2},\, uw_i,\, v_{ii+1}u,\, v_{ii+1}v_{jj+1},\, v_{ii+1}w_j (j \neq i+1) : 1 \le i,j \le n\}$

Use induction to determine $d_k$:

- If $x \in \mathcal{B}$ is a word of length $k-1$, then $w_ix \in \mathcal{B}$. Thus there are $nd_{k-1}$ words of length $k$ in $\mathcal{B}$ starting with $w_i$.
- If $x \in \mathcal{B}$ is a word of length $k-1$, then $v_{ii+1}x \in \mathcal{B}$ if and only if $x$ does not begin with $w_{i+1}$. As determined in the previous bullet, there are $nd_{k-2}$ basis elements starting with $w_j, 1 \le j \le n$ in degree $k-1$, and thus $d_{k-2}$ of them beginning with $w_{i+1}$. Hence, for each $i$, there are $d_{k-1}-d_{k-2}$ possibilities for $x$. Therefore, there are $n(d_{k-1}-d_{k-2})$ words of length $k$ of the form $v_{i\,i+1}x$.
- We will treat the case of words beginning with $u$ in three cases. If $x \in \mathcal{B}$ of length $k-2$, $uux \in \mathcal{B}$ if and only if $x$ does not begin with $v_{ii+1}, 2 \le i \le n$. There are $d_{k-1}-nd_{k-2}-n(d_{k-2}-d_{k-3})$ words beginning with $u$ in degree $k-1$ (from previous bullets). Thus, there are that many words of the form $uux \in \mathcal{B}$. Next $uv_{12}x \in \mathcal{B}$ if and only if $x$ does not begin with $w_i, 2 \le i \le n$. Thus, there are $d_{k-2}-(n-1)d_{k-3}$ words of the form $uv_{12}x$. Finally, $uw_ix \in \mathcal{B}$ for all $x$. Thus, there are $nd_{k-2}$ words of this form. This gives us a total of $d_{k-1}-2nd_{k-2}+nd_{k-3}+d_{k-2}-(n-1)d_{k-3}+nd_{k-2} = d_{k-1}-(n-1)d_{k-2}+d_{k-3}$ words beginning with $u$.

Thus, $d_k = nd_{k-1}+n(d_{k-1}-d_{k-2})+d_{k-1}-(n-1)d_{k-2}+d_{k-3} = (2n+1)d_{k-1}-(2n-1)d_{k-2}+d_{k-3}$.

We can write this recurrence formula as a generating function following the method described by Wilf in [[Wilf],§1.2]. Let $H(t) = \sum_{i\geq 0} d_i t^i$ denote the generating function that we are trying to find. Let $d_{-2} = d_{-1} = 0, d_0 = 1$. Multiply both sides of the recursion by $t^i$ and sum over $i \geq 0$. Then on the left-hand side we have $d_1 + d_2 t + d_3 t^2 + ... = \frac{H(t)-d_0}{t}$. And on the right hand side we have $(2n+1)H(t) - (2n-1)tH(t) + t^2 H(t)$. Solving for $H(t)$:

$H(t) - 1 = H(t)[(2n+1)t - (2n-1)t^2 + t^3] \Rightarrow$

$H(t)[1 - (2n+1)t + (2n-1)t^2 - t^3] = 1 \Rightarrow$

$H(t) = \frac{1}{1-(2n+1)t+(2n-1)t^2-t^3}$.

**Method 2:** [[RSW],Thm 2] gives the Hilbert series formula as:

$$H(A(\Gamma), t) = \frac{1-t}{1 + \sum_{v_1 > \cdots > v_l \geq *} (-1)^l t^{|v_1| - |v_l| + 1}}. \tag{1}$$

In this example, the possible sequences indexing the sum are: $u, v_{ii+1}, w_i, *, u > v_{ii+1}, v_{ii+1} > w_i, v_{ii+1} > w_{i+1}, w_i > *, u > w_i, u > v_{ii+1} > w_i, u > v_{ii+1} > w_{i+1}, v_{ii+1} > *, v_{ii+1} > w_i > *, v_{ii+1} > w_{i+1} > *, u > *, u > v_{ii+1} > *, u > w_i > *, u > v_{ii+1} > w_i > *$, and $u > v_{ii+1} > w_{i+1} > *$. Thus, the coefficients of $t$, $t^2$, $t^3$, and $t^4$ are $-(2n+2)$, $n + 2n + n = 4n$, $n + n - 2n - 2n = -2n$, and $1 - 2n + 2n = 1$, respectively. The coefficient of $t^k$ for $k \geq 5$ is zero. Thus

$$H(A(\Gamma_{D_n}), t) = \frac{1-t}{1-(2n+2)t+4nt^2-2nt^3+t^4}. \qquad \square$$

## 3.2. The Algebra $Q_n$

The algebras $Q_n$ are the algebras associated with the lattice of subsets of $\{1, 2, ..., n\}$. Label the vertices in level $i$ by $\{v_A : A \subseteq \{1, ..., n\}, |A| = i\}$. $Q_4$ is shown in Figure 2 below. Their history and some properties are discussed in [GRSW].

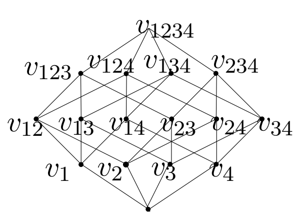


FIGURE 2. $Q_4$

Following the definition given in Proposition 1.2, the generators of $Q_n$ are the vertices $\{v_A : A \subseteq \{1, ..., n\}\}$ and the relations are

$$(*)\{v_A(v_{A\setminus i} - v_{A\setminus j}) - v_{A\setminus i}^2 + v_{A\setminus j}^2 + (v_{A\setminus i} - v_{A\setminus j})v_{A\setminus\{i,j\}} : A \subseteq \{1, ..., n\}, i, j \in A\}.$$

Furthermore,

**Proposition 3.4.** *$\mathcal{B}_{\mathcal{Q}} = \{e(v_{A_1}, k_1) \cdots e(v_{A_l}, k_l) : l \geq 0, A_1, ..., A_l \subseteq \{1, ..., n\}, 1 \leq k_i \leq |v_{A_i}| = |A_i|, (v_{A_i}, k_i) \not\gtrdot (v_{A_{i+1}}, k_{i+1})\}$ is a basis for $Q_n$.*

*Proof.* This follows directly from Theorem 1.1. □

In [[RSW],Thm 3], Retakh, Serconek, and Wilson prove that

$$H(Q_n, t) = \frac{1 - t}{1 - t(2 - t)^n}$$

using Equation 1.

## 4. Automorphism Groups of the Algebras

Throughout this paper Aut$(A)$ will denote the filtration-preserving automorphisms of the graded algebra $A$ (see § 1).

**Lemma 4.1.** *$Aut(A(\Gamma)) \supseteq k^* \times Aut(\Gamma)$*

*Proof.* Any automorphism $\tilde{\sigma}$ of the graph extends to an automorphism $\sigma$ of $T(E)$. Since $\tilde{\sigma}$ preserves paths, $\sigma$ preserves the ideal $R$ defined in Section 1. Hence it induces an automorphism, again denoted by $\sigma$, of $A(\Gamma) = T(E)/R$. Also, for any scalar $\alpha$, multiplication by $\alpha$ is an automorphism because the relations are homogeneous. Thus $Aut(A(\Gamma)) \supseteq k^* \times \mathrm{Aut}(\Gamma)$. □

Let $\Gamma$ be a graph with a unique minimal vertex at level 0, $V_0 = \{*\}$, whose vertices are labeled in the following manner. Label the vertices in level one by $\{v_1, ..., v_m\}$ and index those in level $r$, $2 \leq r \leq n$, by a subset of the power set of $\{1, ..., m\}$. Let the edges connect vertices by minimal containment of their indices. There is a path from $v_A$ ($|v_A| > |v_i|$) to $v_i$ if and only if $i \in A$.

**Theorem 4.2.** *Let $\Gamma$ be a graph as described above. If $\Gamma$ satisfies i) $|V_1| > 2$, ii) no two vertices have the same label, and iii) there are either zero or two paths between any two vertices which are two levels apart in $\Gamma$, then Aut($A(\Gamma)$) $= k^* \times$ Aut$(\Gamma)$, $k$ the base field.*

*Proof.* By Lemma 4.1, $Aut(A(\Gamma)) \supseteq k^* \times \mathrm{Aut}(\Gamma)$. Any automorphism of the algebra must preserve the relations. Thus, for all subsets $B, C \subseteq A \subseteq \{1, ..., m\}$ such that $v_A, v_B, v_C \in \Gamma$, $|v_A| \geq 2$, and $|v_A| - |v_B| = |v_A| - |v_C| = 1, |v_A| - |v_{B\cap C}| = 2$ (i.e. $v_A, v_B, v_C, v_{B\cap C}$ form a diamond), the image of $v_A(v_B - v_C) - v_B^2 + v_C^2 + (v_B - v_C)v_{B\cap C}$ must equal zero. Consider first paths from level 2 to level 0. Because of our assumption that there are exactly two paths from each vertex in level two, $|A| = 2$ for $v_A \in V_2$. Let $\sigma \in Aut(A(\Gamma))$ and $v_{A_1}, ..., v_{A_{m_2}}$ be vertices in level 2; then $\sigma(v_{ij}) = a_{A_1}^{ij} v_{A_1} + \cdots + a_{A_{m_2}}^{ij} v_{A_{m_2}} + b_1^{ij} v_1 + \cdots + b_m^{ij} v_m$ for all $i, j$ such that $v_{ij} \in \Gamma$ and $\sigma(v_i) = c_1^i v_1 + \cdots + c_m^i v_m$ for all $i$, where all coefficients are in k.

Now $\sigma(v_{ij}(v_i - v_j)) = \sigma(v_i^2 - v_j^2)$ implies
$(a^{ij}_{A_1} v_{A_1} + \cdots + a^{ij}_{A_{m_2}} v_{A_{m_2}} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)((c^i_1 - c^j_1)v_1 + \cdots + (c^i_m - c^j_m)v_m)$
$= (c^i_1 v_1 + \cdots + c^i_m v_m)^2 - (c^j_1 v_1 + \cdots + c^j_m v_m)^2$.
There are no $v'_A s$ with $|A| = 2$ on the right-hand side, and so we must use our relations to eliminate them from the left-hand side. Thus, every occurrence of $v_{kl}$ must be followed by $v_k - v_l$; and hence, $c^i_k - c^j_k = -(c^i_l - c^j_l)$ and $c^i_m - c^j_m = 0$ if $m \neq k, l$. Therefore, if $a^{ij}_{kl} \neq 0$,

$$(c^i_1 - c^j_1)v_1 + \cdots + (c^i_m - c^j_m)v_m = (c^i_k - c^j_k)(v_k - v_l). \tag{2}$$

This has two consequences. First, at most one $a^{ij}_{kl}$ can be nonzero. If all $a^{ij}_{kl}$ were zero, the element $v_{i\,j} \notin (A(\Gamma))_{(1)}$ would be sent to an element in $(A(\Gamma))_{(1)}$, which we cannot allow because then $\sigma$ would not be invertible. Thus $a^{ij}_{kl}$ must be nonzero for exactly one $\{kl\}$. Let us denote this set by $\{\tau(i)\tau(j)\}$. Then $\sigma(v_{ij}) = a^{ij}_{\tau(i)\tau(j)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m$. If $\tau(ij) = \tau(kl)$, then $\sigma(v_{i\,j} - a^{ij}_{\tau(i)\tau(j)}(a^{kl}_{\tau(k)\tau(l)})^{-1} v_{k\,l}) \in (A(\Gamma))_{(1)}$; this implies that $\{ij\} = \{kl\}$. Thus $\tau$ is one-to-one, and so is in $S_n$.

A second consequence of (2) is that $c^i_r - c^j_r$ is zero if and only if $r \neq \tau(i), \tau(j)$.

We now have from (2): $(a^{ij}_{\tau(ij)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) =$
$(c^i_1 v_1 + \cdots + c^i_m v_m)^2 - (c^j_1 v_1 + \cdots + c^j_m v_m)^2$. (Recall $c^i_{\tau(i)} - c^j_{\tau(i)} = -(c^i_{\tau(j)} - c^j_{\tau(j)})$.)

Let $z = \sum_{r \neq \tau(i), \tau(j)} c^i_r v_r = \sum_{r \neq \tau(i), \tau(j)} c^j_r v_r$. Then

$$\begin{aligned}
&(a^{ij}_{\tau(ij)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)((c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) \\
&= (c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)} + z)^2 - (c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)} + z)^2 \\
&= (c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)})^2 - (c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)})^2 + (c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)})z \\
&+ z(c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)}) - (c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)})z - z(c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)}).
\end{aligned} \tag{3}$$

On the left-hand side of (3), $v_r$, for $r \neq \tau(i), \tau(j)$, is never the second term of the product of two $v'_r s$. Hence, $(c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)} - c^j_{\tau(i)} v_{\tau(i)} - c^j_{\tau(j)} v_{\tau(j)})z = 0$. This implies that either $c^i_{\tau(i)} = c^j_{\tau(i)}$ and $c^i_{\tau(j)} = c^j_{\tau(j)}$, which is a contradiction since $\sigma(v_i) \neq \sigma(v_j)$, or $z = 0$. Thus $z = 0$ and so $c^i_r = c^j_r = 0$ for all $r \neq \tau(i), \tau(j)$. Now we have
$(a^{ij}_{\tau(ij)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) \equiv$
$a^{ij}_{\tau(ij)}(c^i_{\tau(i)} - c^j_{\tau(i)})(v^2_{\tau(i)} - v^2_{\tau(j)}) + (b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) =$
$((c^i_{\tau(i)})^2 - (c^j_{\tau(i)})^2)v^2_{\tau(i)} + (c^i_{\tau(i)} c^i_{\tau(j)} - c^j_{\tau(i)} c^j_{\tau(j)})(v_{\tau(i)} v_{\tau(j)} + v_{\tau(j)} v_{\tau(i)}) + ((c^i_{\tau(j)})^2 - (c^j_{\tau(j)})^2)v^2_{\tau(j)}$

Because the right-hand side is in the subspace generated by $v_{\tau(i)}, v_{\tau(j)}$, the left-hand side is as well. Therefore, only $b^{ij}_{\tau(i)}, b^{ij}_{\tau(j)}$ can be nonzero.

Let us write down what we know so far. For any $i, j$, $1 \leq i, j \leq n$, we have:
1) $\sigma(v_{i\,j}) = a^{ij}_{\tau(ij)} v_{\tau(i)\tau(j)} + b^{ij}_{\tau(i)} v_{\tau(i)} + b^{ij}_{\tau(j)} v_{\tau(j)}$
2) $\sigma(v_i) = c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)}$
3) $\sigma(v_j) = c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)}$
and

4) $(a^{ij}_{\tau(ij)}v_{\tau(i)\tau(j)} + b^{ij}_{\tau(i)}v_{\tau(i)} + b^{ij}_{\tau(j)}v_{\tau(j)})(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) \equiv$
$a^{ij}_{\tau(ij)}(c^i_{\tau(i)} - c^j_{\tau(i)})(v^2_{\tau(i)} - v^2_{\tau(j)}) + (b^{ij}_{\tau(i)}v_{\tau(i)} + b^i_{\tau(j)}v_{\tau(j)})(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) =$
$((c^i_{\tau(i)})^2 - (c^j_{\tau(i)})^2)v^2_{\tau(i)} + (c^i_{\tau(i)}c^i_{\tau(j)} - c^j_{\tau(i)}c^j_{\tau(j)})(v_{\tau(i)}v_{\tau(j)} + v_{\tau(j)}v_{\tau(i)}) + ((c^i_{\tau(j)})^2 - (c^j_{\tau(j)})^2)v^2_{\tau(j)}$

Applying (1) and (3) above to $\{ik\}$ (we are using here that $|V_1| > 2$) we find that $\sigma(v_i) = c^i_{\tau(i)}v_{\tau(i)} + c^i_{\tau(k)}v_{\tau(k)}$. Because $\sigma(v_i) \neq 0$, $c^i_{\tau(i)}$ and $c^i_{\tau(j)}$ cannot both be zero. Furthermore, $\tau(i)$, $\tau(j)$, and $\tau(k)$ are distinct, so $c^i_{\tau(i)} \neq 0$ and $c^i_{\tau(j)} = c^i_{\tau(k)} = 0$. Thus, $\sigma(v_i) = c^i_{\tau(i)}v_{\tau(i)}$.

Because we have $c^i_{\tau(i)} - c^j_{\tau(i)} = -(c^i_{\tau(j)} - c^j_{\tau(j)})$ and $c^i_{\tau(j)} = 0 = c^j_{\tau(i)}$, we see that $c^i_{\tau(i)}$ is independent of $i$; call this coefficient c. Thus,
$a^{ij}_{\tau(ij)}c(v^2_{\tau(i)} - v^2_{\tau(j)}) + (b^{ij}_{\tau(i)}v_{\tau(i)} + b^{ij}_{\tau(j)}v_{\tau(j)})c(v_{\tau(i)} - v_{\tau(j)}) = c^2(v^2_{\tau(i)} - v^2_{\tau(j)})$
$\Rightarrow b^{ij}_{\tau(i)} = b^{ij}_{\tau(j)} = 0$ and $a^{ij}_{\tau(ij)} = c$ for all $\{ij\}$.

What $\sigma$ does on level one forces what happens on the levels above. We may compose $\sigma$ with the automorphism that multiplies each element by $1/c$; call this composition $\hat{\sigma}$. We have shown that $\hat{\sigma}$ permutes the vertices in levels 1 and 2. Assume that $\hat{\sigma}$ permutes the vertices in levels less than or equal to $k-1$;i.e. $\hat{\sigma}(v_B) = v_{\tau(B)}$. Let $v_A, v_{A_1}, ..., v_{A_{m_k}} \in V_k$, $v_B, v_C, v_{B_1}, ..., v_{B_{m_{k-1}}} \in V_{k-1}$. Each vertex $v_A$ in level $k$ is present in at least one relation $v_A(v_B - v_C) - v_B^2 + v_C^2 + (v_B - v_C)v_{B\cap C} = 0$. Apply $\hat{\sigma}$ to this relation and we get $(a^A_{A_1}v_{A_1} + \cdots + a^A_{A_{m_k}}v_{A_{m_k}} + b^A_{B_1}v_{B_1} + \cdots + b^A_{B_{m_k-1}}v_{B_{m_{k-1}}})(v_{\tau(B)} - v_{\tau(C)}) - v^2_{\tau(B)} + v^2_{\tau(C)} + (v_{\tau(B)} - v_{\tau(C)})v_{\tau(B\cap C)}$ by the induction hypothesis. In order for this to equal 0, we must have that $v_A$ goes to $v_{\tau(B)\cup\tau(C)} = v_{\tau(B\cup C)} = v_{\tau(A)}$. Thus, $\hat{\sigma}(v_A) = v_{\tau(A)}$ for all $v_A \in V$, and so $\tau \in$Aut$(\Gamma)$.

Therefore, Aut$(A(\Gamma)) = k^* \times$ Aut$(\Gamma)$. □

**Lemma 4.3.** *Aut*$(\Gamma_{D_n}) = D_n$.

*Proof.* Any automorphism of the graph must preserve the set of vertices at each level and so acts on the set $\{w_1, ..., w_n\}$ of all n vertices in level 1. We may say $\sigma(w_i) = w_{\sigma(i)}$ (slightly abusing the use of $\sigma$). Thus we can think of an automorphism of the graph as being a permutation in $S_n$ acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels; i.e. $\sigma(v_{ij}) = v_{\sigma(i)\,\sigma(j)}$. Labeling the vertices in level two by the vertices they are connected to in level one ensures that as long as the set of vertices in each level is preserved, the edges will be as well.

Recall that $V_2$ refers to the vertices in level two of the graph. Only permutations which send the set $V_2 = \{(i\,i+1) : 1 \leq i \leq n\}$ to itself are allowed. Clearly $r = (12...n)$ fixes $V_2$. We may replace $\sigma$ by $r^i\sigma$ for some $i$ and assume $\sigma(1) = 1$. Then $\sigma(12)$ is either $(12)$ or $(1n)$, which implies either $\sigma(2) = 2$ (and thus $\sigma = id$) or $\sigma(2) = n$. In the latter case $\sigma = (2n)(3\,n-1)(4n-2)\cdots = s$. Thus $r$ and $s$ generate the automorphism group of $\Gamma_{D_n}$; this is the dihedral group on n elements, $D_n$. Note that these automorphisms may be viewed as reflections and rotations of the n-gon. □

**Theorem 4.4.** *a) If $n \geq 3$, Aut$(A(\Gamma_{D_n})) = k^* \times D_n$, k the base field*

*b) If $n = 2$,*

$$Aut(A(\Gamma_{D_2})) \cong \{M \in GL(3,k) : M = \begin{bmatrix} c^1_1 + c^1_2 & c^2_1 - c^1_2 & c^1_2 - c^2_1 \\ 0 & c^1_1 & c^1_2 \\ 0 & c^2_1 & c^1_1 + c^1_2 - c^2_1 \end{bmatrix}, c^j_i \in k\,\forall i,j\}$$

*Proof.* a) By Lemma 4.3, $\text{Aut}(\Gamma_{D_n}) = D_n$. It is clear by looking at the graph $\Gamma_{D_n}$ (Figure 1) that for $n > 2$ $\Gamma_{D_n}$ satisfies the conditions of Theorem 4.2. Therefore, $\text{Aut}(A(\Gamma_{D_n})) = k^* \times D_n$.

b) In this case, $\Gamma_{D_2}$ fails to satisfy condition (i) of Theorem 4.2; there are only two vertices on level 1. Consider the proof of Theorem 4.2. The proof is valid up until we apply (1) and (3) to $\{ik\}$ to find that $\sigma(v_i) = c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(k)} v_{\tau(k)}$. In the case where $n = 2$, $\tau(i) + 1 = \tau(i-1) + 2 = \tau(i-1)$ in $\mathbb{Z}/(2)$, so $c^i_{\tau(i)+1} = c^i_{\tau(i-1)}$ can be nonzero. Thus, $w_i$ can go to a sum of multiples of $w_1$ and $w_2$; $\sigma(w_i) = c^i_1 w_1 + c^i_2 w_2$. Because we only have one vertex in level two, $v_{12}$, it can only go to a multiple of itself plus multiples of $w_1$ and $w_2$. Thus we can drop the sub and superscripts on $a$ and the superscripts on $b_i$: $\sigma(v_{12}) = a v_{12} + b_1 w_1 + b_2 w_2$. We can rewrite (4) in the proof of Theorem 4.2 as $a(c^1_1 - c^2_1)(w_1^2 - w_2^2) + (b_1 w_1 + b_2 w_2)(c^1_1 - c^2_1)(w_1 - w_2) = ((c^1_1)^2 - (c^2_1)^2) w_1^2 + (c^1_1 c^1_2 - c^2_1 c^2_2)(w_1 w_2 + w_2 w_1) + ((c^1_2)^2 - (c^2_2)^2) w_2^2$. We can conclude from this that $a + b_1 = c^1_1 + c^2_1$, $a + b_2 = c^1_2 + c^2_2$, and $-b_1(c^1_1 - c^2_1) = c^1_1 c^1_2 - c^2_1 c^2_2 = b_2(c^1_1 - c^2_1) \Rightarrow -b_1 = b_2$ (else $c^1_1 c^1_2 = c^2_1 c^2_2 \Rightarrow c^1_1 = c^2_1$ and $c^1_2 = c^2_2$, which is not possible). These imply that $2a = c^1_1 + c^1_2 + c^2_1 + c^2_2 \Rightarrow a = c^1_1 + c^1_2 \Rightarrow b_1 = c^2_1 - c^1_2 = -b_2$.

Write the element $r v_{12} + s w_1 + t w_2$ as the vector $[\ r \quad s \quad t\ ]$. Then a way to visualize what this automorphism group looks like is to consider the invertible transformation matrix M that sends $[\ r \quad s \quad t\ ] \mapsto [\ r \quad s \quad t\ ] * M$

$$M = \begin{bmatrix} c^1_1 + c^1_2 & c^2_1 - c^1_2 & c^1_2 - c^2_1 \\ 0 & c^1_1 & c^1_2 \\ 0 & c^2_1 & c^1_1 + c^1_2 - c^2_1 \end{bmatrix}$$

This matrix is conjugate to a triangular matrix and thus stabilizes a flag. The spaces $M$ stabilizes can be found by solving $[\ r \quad s \quad t\ ] * M = \alpha [\ r \quad s \quad t\ ]$. $M$ stabilizes the one-dimensional spaces $k \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$ and $k \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$. □

Denote the lattice of subsets of $\{1, ..., n\}$ by $\mathcal{L}_{[n]}$. In other words, $Q_n = A(\mathcal{L}_{[n]})$.

**Lemma 4.5.** *If $n \geq 3$, $Aut(\mathcal{L}_{[n]}) = S_n$.*

*Proof.* Any automorphism of the graph must preserve the set of vertices at each level and so acts on the set $\{v_1, ..., v_n\}$ of all n vertices in level 1; so, we may say $\sigma(v_i) = v_{\sigma(i)}$ (slightly abusing the use of $\sigma$). Thus we can think of an automorphism of the graph as being a permutation in $S_n$ acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels; i.e. $\sigma(v_A) = v_{\sigma(A)}$. Labeling the vertices in levels two and higher by the vertices to which there is a path to in level one ensures that as long as the set of vertices in each level is preserved, the edges will be as well. Since for each subset of $\{1, ..., n\}$ of cardinality $i$ level $i$ has a vertex labeled by that subset, every element of $S_n$ is an automorphism of the graph. In other words, for every $\tau \in S_n$, $\tau$ will permute the vertices on each level. □

**Theorem 4.6.** *If $n \geq 3$, $Aut(Q_n) = k^* \times S_n$.*

*Proof.* We can see that $\mathcal{L}_{[n]}$ satisfies conditions (i) and (ii) of Theorem 4.2 since each subset of $\{1, ..., n\}$ occurs exactly once as a vertex in the lattice and for $n > 2$ there are more than 2 singleton subsets. Condition (iii) is satisfied because each vertex $v_B$ directly below a vertex $v_A$ is obtained by removing exactly one element from $A$. Thus for any vertex $v_C$ two levels below $v_A$,

$|C| = |A| - 2$. Say $C = A \backslash \{i, j\}$. There are only two ways to obtain $C$: first remove $i$ then $j$ or vice versa. Therefore, there are only two paths from $v_A$ to $v_C$.

Therefore, by Lemma 4.5 and Theorem 4.2, $\mathrm{Aut}(Q_n) = k^* \times S_n$. $\square$

Consequently, in both of these algebras, the $\mathrm{Aut}(A(\Gamma))$-submodules of $A(\Gamma)_{[i]}$ are precisely the $\mathrm{Aut}(\Gamma)$-submodules. Since $\mathrm{Aut}(\Gamma)$ is finite we have that $A(\Gamma)_{[i]}$ is a completely reducible $\mathrm{Aut}(A(\Gamma))$-module whenever characteristic $k = 0$.

## 5. Graded trace generating functions

Pass to the associated graded algebra, $\mathrm{gr}A(\Gamma)$. Let $\phi_1, ..., \phi_l$ denote all of the distinct irreducible representations of $\mathrm{Aut}A(\Gamma)$ and let $\chi_j$ denote the character afforded by $\phi_j$. $\mathrm{Aut}A(\Gamma)$ acts on each $A(\Gamma)_{[i]}$, and so the completely reducible $\mathrm{Aut}A(\Gamma)$-module $A(\Gamma)_{[i]}$ may be written as $\bigoplus_{j=1}^{l} m_{ij}\phi_j$. The basis $\mathcal{B}(\Gamma)$ of $A(\Gamma)$ is invariant under the automorphism $\sigma$. Therefore, the trace of $\sigma$ on $grA(\Gamma)$, $Tr\sigma|_{A(\Gamma)}$, is the number of fixed basis elements.

Remark: $Tr\sigma$ is the dimension of the subalgebra $A(\Gamma^\sigma)$, which is not the same as the dimension of the fixed point space. The subalgebra $A(\Gamma^\sigma)$ described in Section 2 is the span of the set of fixed elements of the basis. On the other hand, the fixed point space is the span of the sums of orbits of $\sigma$. Averages over orbits are in the fixed point space, but not in the subalgebra.

We will give two methods by which to find the graded trace generating functions for general $A(\Gamma)$. The first will be to essentially count "allowable" and "non-allowable" words - a generating function that gives the number of irreducible words in each grading in the subalgebra $A(\Gamma^\sigma)$. The second will generalize Equation 1 to use on the subgraph $\Gamma^\sigma$ and subalgebra $A(\Gamma^\sigma)$. These graded trace generating functions will be used to find the multiplicities of irreducible representations.

### 5.1. Method 1 - Counting fixed words:

The $Tr\sigma|_{A(\Gamma)_{[i]}}$ is the number of fixed basis elements of degree i. In other words, the number of sequences $(x_1, k_1), ..., (x_l, k_l)$ such that $1 \le k_j \le |x_j|$, $k_1 + \cdots + k_l = i$, $e(x_i, k_1) \cdots e(x_l, k_l)$ is irreducible and $\sigma x_j = x_j \, \forall j$. Recall that $e(x_i, k)e(x_j, l)$ is reducible if there is a path from $x_i$ to $x_j$ and the level of $x_i$ equals the level of $x_j$ plus k.

**Lemma 5.1.** *Let $X$ be a vector space with fixed basis $\mathcal{B}$. Let $Z = \{V \subseteq X : V$ subspace, $V = span(V \cap \mathcal{B})\}$. Then $Z(+, \cap)$ is a lattice isomorphic to the lattice of $\mathcal{P}(\mathcal{B})(\cap, \cup)$.*

*Proof.* The map $\phi : Z \to \mathcal{P}(\mathcal{B})$ defined by $\phi(V) = V \cap \mathcal{B}$ is a lattice isomorphism. $\square$

Remark: The lattice of subsets of $\mathcal{B}$ is distributive, and so $Z$ is distributive.

**Theorem 5.2.** *Let $X = \sum X_i$ be a graded vector space and let the basis $\mathcal{X}$ of $X$ consist of homogeneous elements; $\mathcal{X} = \cup X_i$, where $\mathcal{X}_i = \mathcal{X} \cap X_i$. Then $T(X)$ is bi-graded with $T(X)_{i,j} =$*

*span*$\{x_{l_1}\cdots x_{l_i} : x_{l_k} \in \mathcal{X}_{l_k}, l_1+\cdots+l_i = j\}$*. Let* $\mathcal{Y}$ *be a finite set of quadratic monomials in* $\mathcal{X}$*, and define* $Y =< \mathcal{Y} >\subseteq T(X)$*. Let* $|\cdot|$ *denote the bi-graded dimension of the space. Then*

$$|T(X)/Y| = \frac{1}{1-|X|+|Y|-|XY\cap YX|+|X^2Y\cap XYX\cap YX^2|-\cdots}.$$

*Proof.* Denote $T(X)/Y$ by $A$ in this proof. The graded dimension of $T(X)$ is $\frac{1}{1-|X|}$. Because each generator in $Y$ is a quadratic monomial, as vector spaces we can identify $A$ with the subspace of $T(X)$ spanned by words not containing a subword in $Y$.

For $i \geq 0$ define $Y^{(i)} := \bigcap_{j=0}^{i} X^jYX^{i-j}$; note that $Y^{(0)} = Y$. It will also be convenient to define $Y^{(-1)} := X$ and $Y^{(-2)} := k$. For $i \geq -2$, let $T_i := T(X)Y^{(i)}/(T(X)YX^{i+2}T(X)\cap T(X)Y^{(i)})$. Also, define $T_{-3} := T(X)/(T(X)XT(X))$.

Now define a map $\phi_i : T_i \to T_{i-1}$ for $i \geq -1$ by, for $a \in T(X)Y^{(i)}$, $a + T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)} \mapsto a + T(X)YX^{i+1}T(X) \cap T(X)Y^{(i-1)}$. Also, define $\phi_{-2} : T_{-2} \to T_{-3}$ by $a + T(X)YT(X) \mapsto a + T(X)XT(X)$. Because $Y^{(i)} = XY^{(i-1)} \cap YX^i \subseteq XY^{(i-1)}$, $T(X)Y^{(i)} \subseteq T(X)Y^{(i-1)}$. Also, $T(X)YX^{i+2}T(X) \subseteq T(X)YX^{i+1}T(X)$ since $X^{i+2}T(X) = X^{i+1}XT(X) \subseteq X^{i+1}T(X)$. Thus, $\phi_i$ is a well-defined map.

Next we will show that the sequence $\cdots \to T_j \to T_{j-1} \to \cdots \to T_{-2} \to T_{-3} \to 0$ is exact. For $i \geq 0$, $\phi_{i-1}(\phi_i(a + T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)})) = a + T(X)YX^iT(X) \cap T(X)Y^{(i-2)}$. Now $a \in T(X)Y^{(i)} \subseteq T(X)Y^{(i-1)} \subseteq T(X)Y^{(i-2)}$. Also, $a \in T(X)Y^{(i)} = T(X)(YX^i \cap \cdots \cap X^iY) \subseteq T(X)YX^i \subseteq T(X)YX^iT(X)$. Thus, the image is 0 and $\mathrm{im}\phi_i \subseteq \ker\phi_{i-1}$.

To show the other inclusion, note that $\ker\phi_{i-1} = \{a + T(X)YX^{i+1}T(X) \cap T(X)Y^{(i-1)} : a \in T(X)YX^iT(X) \cap T(X)Y^{(i-2)}\}$. For $a \in T(X)Y^{(i-1)}$ define $\bar{a} = a + T(X)YX^{i+1}T(X) \cap T(X)Y^{(i-1)}$. Assume that $\bar{a} \in \ker\phi_{i-1}$. Then $a \in T(X)YX^iT(X)$. We want that $a \in T(X)Y^{(i)}$. To show this first observe that $a \in T(X)Y^{(i-1)} \cap T(X)YX^iT(X) = T(X)Y^{(i-1)} \cap (T(X)YX^i + T(X)YX^{i+1}T(X))$. Let $\mathcal{B}$ be the set of all monomials of $\mathcal{X}$; $\mathcal{B}$ is a basis for $T(X)$ and $\mathcal{Y} \subseteq \mathcal{B}$. Consider the set of $X^iYX^j$. Now $X^iYX^j$ is equal to the span of $\mathcal{X}^i\mathcal{Y}\mathcal{X}^j$, and so Lemma 5.1 applies. Therefore, the lattice generated by all the $X^iYX^j$ is distributive. Hence, we have that $T(X)Y^{(i-1)} \cap (T(X)YX^i + T(X)YX^{i+1}T(X)) = T(X)Y^{(i-1)} \cap T(X)YX^i + T(X)Y^{(i-1)} \cap T(X)YX^{i+1}T(X)$. Since $T(X)Y^{(i-1)} \cap T(X)YX^i \subseteq T(X)Y^{(i)}$, $a \in T(X)Y^{(i)}$ and $\bar{a} \in \mathrm{im}\phi_i$. Therefore, $\mathrm{im}\phi_i = \ker\phi_{i-1}$ for $i \geq 0$.

Since $\phi_{-1}$ is homogeneous in degree, $\mathrm{im}\phi_{-1} = A \cap T(X)X = \ker\phi_{-2}$. Furthermore, the image of $\phi_{-2}$ is nonzero and maps to a one-dimensional space, and thus is surjective. Therefore, the sequence is exact.

Finally we will show that $T_i \cong AY^{(i)}$. We will do this by proving that $T(X)Y^{(i)} = AY^{(i)} \oplus T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)}$. $T(X)Y^{(i)} = (A + T(X)YT(X))Y^{(i)} \subseteq AY^{(i)} + T(X)YT(X)Y^{(i)} \subseteq AY^{(i)} + T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)}$. Also, $AY^{(i)} \cap T(X)YX^{i+2}T(X) \subseteq AX^{i+2} \cap T(X)YT(X)X^{i+2} \subseteq (A \cap T(X)YT(X))X^{i+2} = (0)$. Thus, our claim is proved and $T_i \cong AY^{(i)}$. In particular,
$T_0 = T(X)Y/(T(X)YX^2T(X) \cap T(X)Y) \cong AY$,
$T_{-1} = T(X)X/(T(X)YXT(X) \cap T(X)X) \cong AX$,
$T_{-2} = T(X)/T(X)YT(X) \cong A$, and

$T_{-3} = T(X)/T(X)XT(X) \cong k$. Hence, we can write our exact sequence as
$\cdots \to AY^{(j)} \to AY^{(j-1)} \to \cdots \to AY \to AX \to A \to k \to 0$.
Therefore, $1 = |k| = \sum(-1)^i|T_i| = |A| - |A||X| + \sum_{i\geq 0}(-1)^i|A||Y^{(i)}| = |A|(1-|X|+\sum_{i\geq 0}(-1)^i|Y^{(i)}|)$ implies that

$$|A| = \frac{1}{1-|X|+\sum_{i\geq 0}(-1)^i|Y^{(i)}|} \qquad \square$$

Define another grading on $A(\Gamma)$ by $A(\Gamma)_{[[k]]} = \text{span}\{e(v_1,i_1)\cdots e(v_k,i_k) : (v_j,i_j) \not\gtrdot (v_{j+1},i_{j+1})\}$. This induces an increasing filtration on $A(\Gamma)$, $A(\Gamma)_{((k))} = \text{span}\{e(v_1,i_1)\cdots e(v_j,i_j) : j \leq k, (v_j,i_j) \not\gtrdot (v_{j+1},i_{j+1})\}$. Thus as a vector space we can identify $A(\Gamma)$ with its associated graded algebra, $gr'A(\Gamma)$.

**Lemma 5.3.** *Define $W(\Gamma^\sigma) = span\{e(v,k) : l \geq 0, v \in V_\sigma, 1 \leq k \leq |v|\}$ and $R(\Gamma^\sigma)$ the two-sided ideal in $T(W(\Gamma^\sigma))$ generated by $\tilde{R}(\Gamma^\sigma) = span\{e(v,k)e(u,l) : v > u \in V_\sigma, k = |v|-|u|\}$. Then $A(\Gamma^\sigma) = T(W(\Gamma^\sigma))/R(\Gamma^\sigma)$ as a subalgebra of $gr'A(\Gamma)$.*

*Proof.* This follows from the definitions of $A(\Gamma^\sigma)$ and of $gr'A(\Gamma)$. $\square$

$W(\Gamma^\sigma)$ and $R(\Gamma^\sigma)$ satisfy the hypotheses for Theorem 5.2. Therefore,

$$|A(\Gamma^\sigma)| = \frac{1}{1-|W(\Gamma^\sigma)|+|\tilde{R}(\Gamma^\sigma)|-|\tilde{R}(\Gamma^\sigma)W(\Gamma^\sigma)\cap W(\Gamma^\sigma)\tilde{R}(\Gamma^\sigma)|+\cdots}.$$

## 5.2. Method 2 - Generalizing the Hilbert Series Equation (1):

We would like to apply the function $H(A(\Gamma),t)$ (see Equation (1)) to the subalgebras created by our fixed points. Take the subgraph of $\Gamma$ consisting of the points fixed by an automorphism $\sigma$. This generates a subalgebra of gr$A(\Gamma)$ in the way described in Section 2. Thus, we are using Equation (1) with the additional condition that the vertices in the sum are fixed by $\sigma \in Aut(A(\Gamma))$; call this modified formula $Tr_\sigma(A(\Gamma),t)$.

**Theorem 5.4.** *Let $\Gamma$ be a layered graph with unique minimal element $*$ of level 0 and $\sigma$ an automorphism of the graph. Let $\Gamma^\sigma$ be the subgraph of $\Gamma$ with vertices being those fixed by $\sigma$ (as described in Section 2). Denote the Hilbert series of the subalgebra $A(\Gamma^\sigma)$, which is the graded trace function of $\sigma$ acting on $A(\Gamma)$, by $Tr_\sigma(A(\Gamma),t)$ (or $Tr_\sigma(t)$ when $A(\Gamma)$ is clear). Then*

$$Tr_\sigma(A(\Gamma),t) = \frac{1-t}{1-t\displaystyle\sum_{\substack{v_1>\cdots>v_l\geq *\\ v_1,\ldots,v_l\in V_\sigma}}(-1)^{l-1}t^{|v_1|-|v_l|}}. \tag{4}$$

*Proof.* Write $Tr(t)$ for $Tr_\sigma(A(\Gamma),t)$ in this proof. Let $v_1,...,v_l,v,w \in V_\sigma$. Recall that the basis for $A(\Gamma^\sigma)$ is $\mathcal{B}_\sigma = \{e(v_1,k_1)\cdots e(v_l,k_l) : v_1,...v_l \in V_\sigma, 1 \leq k_i \leq |v_i|, e(v_i,k_i) \not\gtrdot e(v_{i+1},k_{i+1})\}$.

For $v \in (V_\sigma)_+$, define $C_v = \displaystyle\bigcup_{k=1}^{|v|} e(v,k)\mathcal{B}_\sigma$, $B_v = C_v \cap \mathcal{B}_\sigma$, $D_v = C_v \backslash B_v$. Then $\mathcal{B}_\sigma = \{*\} \cup \displaystyle\bigcup_{v\in(V_\sigma)_+} B_v$. Let $Tr_v = Tr_\sigma(B_v,t)$, the graded dimension of the span of $B_v$. Then $Tr(t) = 1 +$

$\sum_{v \in (V_\sigma)_+} Tr_v(t)$. We also have $Tr_\sigma(C_v, t) = (t + \cdots + t^{|v|})Tr(t) = t(\frac{t^{|v|}-1}{t-1})Tr(t)$ and, because $D_v = \bigcup_{v>w>*} e(v, |v| - |w|)B_w$, $Tr_\sigma(D_v, t) = \sum_{v>w>*} t^{|v|-|w|}Tr_w(t)$. Thus,

$$Tr_v(t) = t\left(\frac{t^{|v|}-1}{t-1}\right)Tr(t) - \sum_{v>w>*} t^{|v|-|w|}Tr_w(t). \tag{5}$$

This equation may be written in matrix form. Put an order on $V_\sigma$, arrange the elements in decreasing order, and index the elements of vectors and matrices by this ordered set. Let $\vec{Tr}(t)$ denote the column vector with $Tr_v(t)$ in the $v$-position and 0 in the *-position. Let $\vec{s}$ denote the column vector with entry $t^{|v|} - 1$ in the $v$-position, let $\vec{1}$ denote the column vector having 1 as each entry, and let $\zeta(t)$ denote the matrix with the entry in the $(v, w)$-position being $t^{|v|-|w|}$ if $v \geq w$ and 0 otherwise. Then rewriting Equation 5 gives $\zeta(t)\vec{Tr}(t) = \frac{t}{t-1}\vec{s}Tr(t)$.

Now $\zeta(t) - I$ is a strictly upper triangular matrix, and so $\zeta(t)$ is invertible; $\zeta^{-1}(t) = I - (\zeta(t) - I) + (\zeta(t) - I)^2 - \cdots$. Thus the $(v, w)$-entry of $\zeta^{-1}(t)$ is $\sum_{\substack{v=v_1>\cdots>v_l=w\geq *\\ v_1,\ldots,v_l\in V_\sigma}} (-1)^{l+1}t^{|v_1|-|v_l|}$.

Thus we can multiply $\zeta(t)\vec{Tr}(t) = \frac{t}{t-1}\vec{s}Tr(t)$ by $\zeta^{-1}$ and then multiply by $\vec{1}^T$ to obtain $\vec{1}^T\vec{Tr} = Tr(t) - 1 = \frac{t}{t-1}\vec{1}^T\zeta^{-1}(t)\vec{s}Tr(t)$. Solving for $Tr(t)$ we get

$$Tr(t) = \frac{t-1}{t-1-t\vec{1}^T\zeta^{-1}(t)\vec{s}} = \frac{1-t}{1-t\vec{1}^T\zeta^{-1}(t)\vec{1}} = \frac{1-t}{1-t\sum_{\substack{v_1>\cdots>v_l\geq *\\ v_1,\ldots,v_l\in V_\sigma}} (-1)^{l-1}t^{|v_1|-|v_l|}}.$$

□

Remark 1: $\zeta$ is the standard zeta matrix and $\zeta^{-1}$ is the Möbius matrix when $t = 1$.
Remark 2: We will normally apply this method; although, both methods can theoretically be applied in all layered graph algebras.

In general, if $\sigma$ and $\tau$ are conjugate, $\Gamma^\sigma$ and $\Gamma^\tau$ are isomorphic (by the conjugation acting on the subscripts of vertices). Thus, it is enough to find the graded trace functions for any one representative of each conjugacy class.

## 6. Graded Trace Generating Functions for $A(\Gamma_{D_n})$ and $Q_n$

Let us calculate the generating functions for our algebras. First of all, the graded trace of the identity acting on the algebra is the graded dimension of the algebra. We will derive it using the theorems above to show that we get the same result as the Hilbert series given earlier.

### 6.1. The Algebra $A(\Gamma_{D_n})$

We will now find $Tr_\sigma(A(\Gamma_{D_n}), t) = \frac{1}{1-(a_1t+a_2t^2+a_3t^3)}$ using Method 1 in Section 5. Notice that because $\Gamma_{D_n}$ has three levels, $\tilde{R}(\Gamma^{id}_{D_n})^{(i)} = \{0\}$ for $i \geq 2$. $W(\Gamma^{id}_{D_n})$ has basis

$\{e(u,3), e(u,2), e(u,1), e(v_{i\,i+1},2), e(v_{i\,i+1},1), e(w_i,1),\ 1 \le i \le n\}$. Hence $|W(\Gamma^{id}_{D_n})| = (2n+1)t + (n+1)t^2 + t^3$. The reducible words of degree 2 are $e(u,2)e(w_i,1)$, $e(u,1)e(v_{i\,i+1},2)$, $e(u,1)e(v_{i\,i+1},1)$, $e(v_{i\,i+1},1)e(w_i,1)$, $e(v_{i\,i+1},1)e(w_{i+1},1)$, $1 \le i \le n$. The set of these words is a basis for $\tilde{R}(\Gamma^{id}_{D_n})$, so we have $|\tilde{R}(\Gamma^{id}_{D_n})| = 3nt^2 + 2nt^3$. The overlaps of reducible words are $\{e(u,1)e(v_{i\,i+1},1)e(w_i,1),$ $e(u,1)e(v_{i\,i+1},1)e(w_{i+1}) : 1 \le i \le n\}$. This set is a basis for $\tilde{R}(\Gamma^{id}_{D_n})W(\Gamma^{id}_{D_n}) \cap W(\Gamma^{id}_{D_n})\tilde{R}(\Gamma^{id}_{D_n})$ and so $|\tilde{R}(\Gamma^{id}_{D_n})W(\Gamma^{id}_{D_n}) \cap W(\Gamma^{id}_{D_n})\tilde{R}(\Gamma^{id}_{D_n})| = 2nt^3$.
Hence $a_3 = 1-(n+n)+(n+n) = 1$, $a_2 = (1+n)-(n+n+n) = 1-2n$, and $a_1 = 1+n+n = 2n+1$. Thus, we have

$$Tr_{id}(A(\Gamma_{D_n}), t) = \frac{1}{1-((2n+1)t-(2n-1)t^2+t^3)},$$

which agrees with the earlier result.

Now since only u is fixed by $r^i$, $W(\Gamma^{r^i}_{D_n}) = \operatorname{span}\{e(u,3), e(u,2), e(u,1)\}$ and $\tilde{R}(\Gamma^{r^i}_{D_n}) = \{0\}$. Hence,

$$Tr_{r^i}(A(\Gamma_{D_n}), t) = \frac{1}{1-(t+t^2+t^3)}.$$

If $n$ is even, $W(\Gamma^{s}_{D_n})$ has basis $\{e(u,3), e(u,2), e(u,1), e(v_{12},2), e(v_{12},1), e(v_{n/2+1\,n/2+2},2),$ $e(v_{n/2+1\,n/2+2},1)\}$ and $\tilde{R}(\Gamma^{s}_{D_n})$ has basis $\{e(u,1)e(v_{i\,i+1},2),\ e(u,1)e(v_{i\,i+1},1) : i = 1, n/2+1\}$. Also, $W(\Gamma^{rs}_{D_n})$ has basis $\{e(u,3), e(u,2), e(u,1), e(w_2,1), e(w_{n/2+2},1)\}$ and $\tilde{R}(\Gamma^{rs}_{D_n})$ has basis $\{e(u,2)e(w_2,1),\ e(u,2)e(w_{n/2+2},1)\}$. If $n$ is odd, $W(\Gamma^{s}_{D_n})$ has basis $\{e(u,3), e(u,2), e(u,1), e(v_{12},2), e(v_{12},1), e(w_{(n+3)/2},1)\}$ and $\tilde{R}(\Gamma^{s}_{D_n})$ has basis $\{e(u,1)e(v_{12},2),\ e(u,1)e(v_{12},2),\ e(u,2)e(w_{(n+3)/2},1)\}$. Computing each separately we see that

$$Tr_s(A(\Gamma_{D_n}), t) = Tr_{rs}(A(\Gamma_{D_n}), t) = \frac{1}{1-(3t+t^2-t^3)}.$$

Using Method 2 we can get our graded trace generating functions by applying Equation (4) to the subalgebras of $A(\Gamma_{D_n})$. The automorphism $r^i$ only fixes u and the minimal vertex (see Figure 3). Since we have two vertices, no vertices one or two levels apart, and one pair of vertices three levels apart (and only one path between them),

$$Tr_{r^i} = \frac{1-t}{1-(2t-t^4)} = \frac{1-t}{1-t(2-t^3)} = \frac{1}{1-(t+t^2+t^3)}.$$

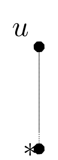


FIGURE 3. $\Gamma^{r}_{D_n}$

The automorphism s acting on the algebra when n is even fixes the top vertex, the minimal vertex, and two vertices on level two ($v_{12}$ and $v_{n/2+1\,n/2+2}$)(see Figure 4). Similarly, when n is odd s fixes the top vertex, the minimal vertex, and one vertex on each of levels one and two ($v_{12}$ and $w_{(n+3)/2}$). Finally, rs fixes the top vertex, the minimal vertex, and two vertices on level one ($w_2$ and $w_{n/2+2}$). Thus, in each case, there are 4 vertices, two edges of length one, and two of length

two. For the coefficient of $t^4$, we have $u > *$, $u >$ vertex $> *$, and $u >$ vertex $> *$. Thus,

$$Tr_s(t) = Tr_{rs}(t) = \frac{1-t}{1-(4t-2t^2-2t^3+t^4)} = \frac{1-t}{1-t(2-t)(2-t^2)} = \frac{1}{1-(3t+t^2-t^3)}.$$

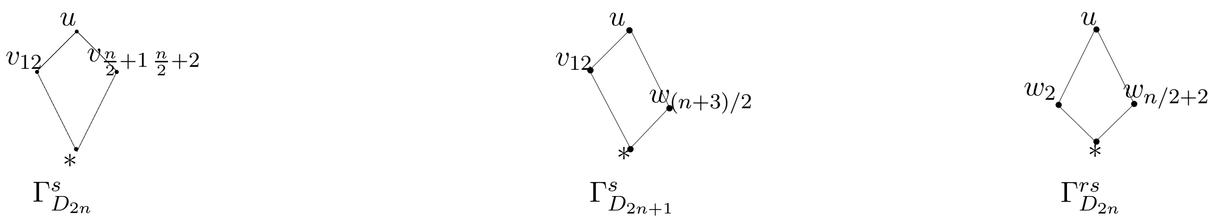


Figure 4. $\Gamma^{\sigma}_{D_n}$

## 6.2. **The Algebra** $Q_n$

Recall that $V_\sigma$ denotes the set of vertices in $\Gamma$ fixed by $\sigma$.

**Theorem 6.1.** *Let $\sigma \in S_n$ and $\sigma = \sigma_1 \cdots \sigma_m$ be its cycle decomposition. Denote the length of $\sigma_j$ by $i_j$ (so $i_j \geq 1$, $i_1 + \cdots + i_m = n$). Then*

$$Tr_\sigma(Q_n, t) = \frac{1-t}{1-t\prod_{j=1}^{m}(2-t^{i_j})}. \tag{6}$$

First of all, notice that when $\sigma = (1)$, this yields $H(Q_n, t)$ given above.

We will prove this using Equation 4 from Section 5:

$$Tr_\sigma(A(\Gamma), t) = \frac{1-t}{1-t\sum_{\substack{v_1 > \cdots > v_l \geq * \\ v_1,\ldots,v_l \in V_\sigma}} (-1)^{l-1} t^{|v_1|-|v_l|}}.$$

If $w \subseteq \{1, ..., n\}$ is $\sigma$-invariant, let $\|w\|$ be the number of $\sigma$-orbits in w. Also, let $\mathcal{O}_j$ denote the non-trivial orbit of $\sigma_j$.

The following Lemma and Corollary and their proofs are parallel to [[RSW],Lemma 2] and [[RSW],Corollary 1]. In the case where $\sigma$ is the identity, they are the same.

**Lemma 6.2.** *Let $w \subseteq \{1, ..., n\}$ be fixed by $\sigma$. Then $\sum_{w=w_1 \supset \ldots \supset w_l = \emptyset} (-1)^l = (-1)^{\|w\|+1}$ where the sum is over all chains of $\sigma$-fixed subsets of $w$.*

*Proof.* If $\|w\| = 1$, then $w = \mathcal{O}_j$ for some j. Thus, there are no fixed proper subsets of w, and we get that both sides are equal to 1. Assume the result holds for all sets with $\|\cdot\| < \|w\|$. Then $\sum_{w \supset \cdots \supset w_l = \emptyset} (-1)^l = \sum_{w \supset w_2 \supseteq \emptyset} \sum_{w_2 \supset \cdots \supset w_l = \emptyset} (-1)^l = \sum_{w \supset w_2 \supseteq \emptyset} (-1)^{\|w_2\|}$ by the induction assumption.

Now $\sum_{w \supset w_2 \supseteq \emptyset} (-1)^{\|w_2\|} = \left( \sum_{w \supseteq w_2 \supseteq \emptyset} (-1)^{\|w_2\|} \right) - (-1)^{\|w\|}$. Say $w = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_r$. Then

the number of $w_2$ such that $w_2 \subseteq w$ and $\|w_2\| = i$ is $\binom{r}{i}$. Hence, we have $\sum_{w \supset w_2 \supseteq \emptyset} (-1)^{\|w_2\|} = \left( \sum_{w \supseteq w_2 \supseteq \emptyset} (-1)^{\|w_2\|} \right) - (-1)^{\|w\|} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i + (-1)^{\|w\|+1} = (-1)^{\|w\|+1}$, as desired, since the alternating sum of the binomial coefficients is zero. □

**Corollary 6.3.** *Let* $\{1, ..., n\} \supseteq v \supseteq w$ *be fixed by* $\sigma$. *Then* $\sum_{v=v_1 \supset v_2 \supset \cdots \supset v_l = w} (-1)^l = (-1)^{\|v\| - \|w\| + 1}$ *where the sum is over all chains of* $\sigma$*-fixed subsets.*

*Proof.* Let $w'$ denote the complement of $w$ in $v$. Sets $u$ invariant under $\sigma$ and satisfying $v \supseteq u \supseteq w$ are in one-to-one correspondence with $\sigma$-invariant subsets of $w'$ via the map $u \mapsto u \cap w'$. Thus, $\sum_{v=v_1 \supset v_2 \supset \cdots \supset v_l = w} (-1)^l = \sum_{w' = v'_l \supset \cdots \supset v'_1 = \emptyset} (-1)^l = (-1)^{\|w'\|+1}$ by the lemma. Because $\|w'\| = \|v\| - \|w\|$, this gives us what we want. □

*Proof of Theorem 6.1.* By Corollary 6.3,

$$\sum_{v_1 \supset \cdots \supset v_l \supseteq \emptyset} (-1)^l t^{|v_1| - |v_l| + 1} = \sum_{\{1,...,n\} \supseteq v_1 \supseteq v_l \supseteq \emptyset} (-1)^{\|v_1\| - \|v_l\| + 1} t^{|v_1| - |v_l| + 1}$$
$$= \sum_{\substack{w, v_l \\ w \cap v_l = \emptyset}} (-1)^{\|w\|+1} t^{|w|+1} = \sum_{w} 2^{m - \|w\|} (-1)^{\|w\|+1} t^{|w|+1}$$

The $\sigma$-invariant sets $w$ are unions of $\sigma$-orbits. Write $a_j = 1$ if $\mathcal{O}_j$ is contained in $w$ and $a_j = 0$ if not. Then the m-tuple $\{a_1, ..., a_m\}$ tells us which orbits are contained in $w$. We can then write $\sum_w 2^{m - \|w\|} (-1)^{\|w\|+1} t^{|w|+1}$ as $\sum_{a_1, \ldots, a_m \in \{0,1\}} (-1)^{\sum a_j + 1} 2^{m - \sum a_j} t^{\sum (a_j i_j) + 1}$. This equals

$$- t \sum_{a_1, \ldots, a_m \in \{0,1\}} \prod_{j=1}^{m} (-1)^{a_j} 2^{1 - a_j} t^{a_j i_j}$$
$$= -t \prod_{j=1}^{m} \sum_{a_j = 0}^{1} (-1)^{a_j} 2^{1 - a_j} t^{a_j i_j}$$
$$= -t \prod_{j=1}^{m} (2 - t^{i_j})$$

Therefore, we have (6). □

**Example 6.1.** *Here are the graded trace functions for* $Q_4$*:*

$$Tr_{(1)}(Q_4, t) = \frac{1-t}{1 - t(2-t)^4} \qquad Tr_{(12)}(Q_4, t) = \frac{1-t}{1 - t(2-t^2)(2-t)^2}$$
$$Tr_{(123)}(Q_4, t) = \frac{1-t}{1 - t(2-t^3)(2-t)} \qquad Tr_{(12)(34)}(Q_4, t) = \frac{1-t}{1 - t(2-t^2)^2}$$
$$Tr_{(1234)}(Q_4, t) = \frac{1-t}{1 - t(2-t^4)}$$

## 7. Representations of Aut$(A(\Gamma))$ acting on $A(\Gamma)$

Now let us determine the multiplicities of the irreducible representations. Assume $A(\Gamma)_{[i]}$ is a completely reducible Aut$(\Gamma)$-module. Note that this is true in our examples. Fix n. Let the graded trace generating function be denoted by $Tr_\sigma(t) = \sum_i Tr_{\sigma,i}t^i$ where $Tr_{\sigma,i} = Tr\sigma|_{A(\Gamma)_{[i]}}$. Let $\phi$ be an irreducible representation of Aut$(\Gamma)$ and $m_\phi(t) = \sum_i m_{\phi,i}t^i$ where $m_{\phi,i}$ is the multiplicity of $\phi$ in $A(\Gamma)_{[i]}$. Finally, let the matrix $C = [\chi_{\sigma\phi}]$ where $\chi_{\sigma\phi}$ is the trace of $\sigma$ on the module which affords the irreducible representation $\phi$; i.e. $C$ is the character table of Aut$(\Gamma)$.

Then, if we fix the degree, $Tr_{\sigma,i} = \sum_\phi \chi_{\sigma\phi}m_{\phi,i}$; so we have $Tr_\sigma(t) = \sum_\phi \chi_{\sigma\phi}m_\phi(t)$. Write $\vec{Tr}(t) = [Tr_{\sigma_1}(t)...Tr_{\sigma_l}(t)]^T$ and $\vec{m}(t) = [m_{\phi_1}(t)...m_{\phi_l}(t)]^T$. Finally,

$$\vec{Tr}(t) = C^T\vec{m}(t) \implies \vec{m}(t) = (C^T)^{-1}\vec{Tr}(t).$$

### 7.1. Representations of Aut$(A(\Gamma_{D_n}))$ acting on $A(\Gamma_{D_n})$

Recall that the character table for $D_n$ where $n = 2m$ is even is:

| | 1 | r | ... | $r^j$ | ... | $r^m$ | s | rs |
|---|---|---|---|---|---|---|---|---|
| $\chi_{triv}$ | 1 | 1 | ... | 1 | ... | 1 | 1 | 1 |
| $\chi_{1-1}$ | 1 | 1 | ... | 1 | ... | 1 | -1 | -1 |
| $\chi_{-11}$ | 1 | -1 | ... | $(-1)^j$ | ... | $(-1)^m$ | 1 | -1 |
| $\chi_{-1-1}$ | 1 | -1 | ... | $(-1)^j$ | ... | $(-1)^m$ | -1 | 1 |
| $\chi_k$ | 2 | $2\cos(2\pi k/n)$ | ... | $2\cos(2\pi kj/n)$ | ... | $2\cos(2\pi km/n)$ | 0 | 0 |

where $(1 \le k \le m-1)$, $r = (12...n)$, and $s = (12)(3n)(4\,n-1)...(\frac{n}{2}+1\ \frac{n}{2}+2)$; so, $rs = (13)(4n)...(\frac{n}{2}+1\ \frac{n}{2}+3)$.

When $n = 2m+1$ is odd the character table is:

| | 1 | r | ... | $r^j$ | ... | $r^m$ | s |
|---|---|---|---|---|---|---|---|
| $\chi_{triv}$ | 1 | 1 | ... | 1 | ... | 1 | 1 |
| $\chi_{1-1}$ | 1 | 1 | ... | 1 | ... | 1 | -1 |
| $\chi_k$ | 2 | $2\cos(2\pi k/n)$ | ... | $2\cos(2\pi kj/n)$ | ... | $2\cos(2\pi km/n)$ | 0 |

where $(1 \le k \le m)$, $r = (12...n)$ and $s = (12)(3n)...(\frac{n+1}{2}\ \frac{n+5}{2})$.

**Proposition 7.1.** *Let $\vec{m}(t)$ be the vector of the graded multiplicities of the irreducible representations of $D_n$ as described above. Set*
$a = \frac{1}{1-((2n+1)t-(2n-1)t^2+t^3)}$, $b = \frac{1}{1-(t+t^2+t^3)}$, *and* $c = \frac{1}{1-(3t+t^2-t^3)}$.

*a) Let $n$ be even. Then,*

$$\vec{m}(t) = \begin{bmatrix} \frac{1}{2n}a + \frac{n-1}{2n}b + \frac{1}{2}c \\ \frac{1}{2n}a + \frac{n-1}{2n}b - \frac{1}{2}c \\ \frac{1}{2n}(a-b) \\ \frac{1}{2n}(a-b) \\ \frac{1}{n}(a-b) \\ \vdots \\ \frac{1}{n}(a-b) \end{bmatrix}$$

*b) Let $n$ be odd. Then,*

$$\vec{m}(t) = \begin{bmatrix} \frac{1}{2n}a + \frac{n-1}{2n}b + \frac{1}{2}c \\ \frac{1}{2n}a + \frac{n-1}{2n}b - \frac{1}{2}c \\ \frac{1}{n}(a-b) \\ \vdots \\ \frac{1}{n}(a-b) \end{bmatrix}$$

*This is obtained from deleting the third and fourth entries in the $n$ is even case.*

*Proof.* We verify this claim by multiplying the transpose of the character table of $D_n$ by $\vec{m}(t)$. The result is

$$\vec{Tr}(t) = \begin{bmatrix} a \\ b \\ \vdots \\ b \\ c \\ c \end{bmatrix}$$

as desired. □

Notice that all of the representations are realized; and, with large multiplicity.

## 7.2. **Representations of $S_n$ acting on $Q_n$**

Unlike the $A(\Gamma_{D_n})$ case, we cannot write down one table giving all of the values in terms of the graded trace functions. However, we can give them in terms of the Frobenius formula. First, however, we will give an example.

**Example 7.1.** *Irreducible Representations for $Q_4$:*

*The character table for $S_4$ is:*

| | (1) | (12) | (123) | (1234) | (12)(34) |
|---|---|---|---|---|---|
| $\chi_{triv}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_{sgn}$ | 1 | -1 | 1 | -1 | 1 |
| $\chi_3$ | 2 | 0 | -1 | 0 | 2 |
| $\chi_{reg}$ | 3 | 1 | 0 | -1 | -1 |
| $\chi_{sgn\otimes reg}$ | 3 | -1 | 0 | 1 | -1 |

*Let* $a = Tr_{(1)}(Q_4,t) = \frac{1-t}{1-t(2-t)^4}$ $\quad b = Tr_{(12)}(Q_4,t) = \frac{1-t}{1-t(2-t^2)(2-t)^2}$,
$c = Tr_{(123)}(Q_4,t) = \frac{1-t}{1-t(2-t^3)(2-t)}$ $\quad d = Tr_{(1234)}(Q_4,t) = \frac{1-t}{1-t(2-t^4)}$
$e = Tr_{(12)(34)}(Q_4,t) = \frac{1-t}{1-t(2-t^2)^2}$

*Then, the multiplicities of the irreducible representations of $S_4$ acting on $Q_4$ as sums of the graded trace generating functions are:*

$m_{triv} = \frac{1}{24}a + \frac{1}{4}b + \frac{1}{3}c + \frac{1}{4}d + \frac{1}{8}e$
$m_{sgn} = \frac{1}{24}a - \frac{1}{4}b + \frac{1}{3}c - \frac{1}{4}d + \frac{1}{8}e$
$m_3 = \frac{1}{12}a - \frac{1}{3}c + \frac{1}{4}e$

$m_{reg} = \frac{1}{8}a + \frac{1}{4}b - \frac{1}{4}d - \frac{1}{8}e$
$m_{sgn\otimes reg} = \frac{1}{8}a - \frac{1}{4}b + \frac{1}{4}d - \frac{1}{8}e$
*The numerical values for the first few degrees are given below:*

| | $\chi_{triv}$ | $\chi_{sgn}$ | $\chi_3$ | $\chi_{reg}$ | $\chi_{sgn\times reg}$ |
|---|---|---|---|---|---|
| $m_{\phi,1}$ | *4* | *0* | *1* | *3* | *0* |
| $m_{\phi,2}$ | *26* | *1* | *17* | *36* | *13* |
| $m_{\phi,3}$ | *219* | *54* | *239* | *434* | *273* |

We can also write the multiplicities in terms of Frobenius' formula. First we will give the notation used in the formula. Let $C_i$ be a representative from the conjugacy class $i$ and $i_j$ be the number of j-cycles in $i$. Also, let $\lambda$ be a partition of n (representing an irreducible representation), $\Delta(x) = \prod_{i<j}(x_i - x_j)$ and $P_j(x) = x_1^j + \cdots + x_k^j$ where $k$ is at least the number of rows in $\lambda$. Set $l_1 = \lambda_1 + k - 1$, $l_2 = \lambda_2 + k - 2, ..., l_k = \lambda_k$. Finally, for $f(x) \in \mathbb{C}[x_1, ..., x_k]$ the scalar $f(x)_{l_1,...,l_k}$ is defined by $f(x) = \sum_{l_1,...,l_k} f(x)_{l_1,...,l_k} x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}$. Then Frobenius' formula says $\chi_\lambda(C_i) = [\Delta(x)\prod_j P_j(x)^{i_j}]_{l_1,...,l_k}$.

**Proposition 7.2.** *Let $r$ denote the degree and $\lambda_l$ the irreducible representation. Then*

$$m_{\lambda_l,r} = \left[\frac{1}{n!}\sum_{j=partition\ of\ n} \chi_{\lambda_l}(C_j)|\mathcal{C}(j)|Tr_{(j)}\right]_{(l_1,...,l_k,r)}$$

*Proof.* Let

$$S = \begin{bmatrix} \chi_{\lambda_1}(C_1) & \cdots & \chi_{\lambda_1}(C_k) \\ \vdots & & \vdots \\ \chi_{\lambda_k}(C_1) & \cdots & \chi_{\lambda_k}(C_k) \end{bmatrix}$$

be the character table of $S_n$. By the orthogonality relations,

$$S^T S = D = \begin{bmatrix} \sum_i \chi_{\lambda_i}(C_1)^2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sum_i \chi_{\lambda_i}(C_k)^2 \end{bmatrix}.$$

Thus

$$S^{-1} = D^{-1}S^T = \begin{bmatrix} \chi_{\lambda_1}(C_1)/\sum_i \chi_{\lambda_i}(C_1)^2 & \cdots & \chi_{\lambda_k}(C_1)/\sum_i \chi_{\lambda_i}(C_1)^2 \\ \vdots & & \vdots \\ \chi_{\lambda_1}(C_k)/\sum_i \chi_{\lambda_i}(C_k)^2 & \cdots & \chi_{\lambda_k}(C_k)/\sum_i \chi_{\lambda_i}(C_k)^2 \end{bmatrix}.$$

Because $\vec{m}(t)S = \vec{Tr}(t)$, $\vec{m}(t) = \vec{Tr}(t)S^{-1}$; and so,

$$m_{\lambda_l}(t) = \frac{\chi_{\lambda_l}(C_1)Tr_{\sigma_1}}{\sum_i \chi_{\lambda_i}(C_1)^2} + \cdots + \frac{\chi_{\lambda_l}(C_k)Tr_{\sigma_k}}{\sum_i \chi_{\lambda_i}(C_k)^2}.$$

However, $\sum_i \chi_{\lambda_i}(C_j)^2 = [S_n : \mathcal{C}(j)] = n!/|\mathcal{C}(j)|$, where $|\mathcal{C}(j)|$ is the size of the conjugacy class of partition $j$. Thus,

$$m_{\lambda_l,r} = \left[\frac{1}{n!} \sum_{j=\text{partition of n}} \chi_{\lambda_l}(C_j)|\mathcal{C}(j)|Tr_{(j)}\right]_{(l_1,\ldots,l_k,r)} \qquad \square$$

# 8. Representations of $\mathbf{Aut}(A(\Gamma))$ acting on $A(\Gamma)^!$

We will use the same methodology as for $A(\Gamma)$ to determine the irreducible representations that are realized in $A(\Gamma)^!$. See Section 2 for the definition of the dual. However, we will see that $Tr_\sigma(A(\Gamma)^!, t)$ has negative coefficients and so is not a generating function of a graded dimension, unlike in the case of gr$A(\Gamma)$.

## 8.1. Representations of $\mathbf{Aut}(A(\Gamma_{D_n}))$ acting on $A(\Gamma_{D_n})^!$

**Proposition 8.1.** *The set $\{u^*,\ v^*_{ii+1},\ w^*_i\ 1 \le i \le n,\ ^*,\ u^*v^*_{ii+1}\ 1 \le i \le n-1,\ v^*_{ii+1}w^*_i\ 1 \le i \le n,$ and $u^*v^*_{12}w^*_1\}$ is a basis for the graded dual algebra $A(\Gamma_{D_n})^!$.*

*Proof.* The generators of $A(\Gamma_{D_n})^!$ are $u^*$, $v^*_{ii+1}$, $w^*_i$ $1 \le i \le n$. In the associated graded algebra the relations are $v_{ii+1}(w_i - w_{i+1})$ and $u(v_{ii+1} - v_{i+1i+2})$. Thus, the relations in the dual are $u^{*2}, u^*w^*_i, v^*_{ii+1}u^*,\ w^*_iu^*, w^*_iw^*_j, v^*_{ii+1}v^*_{jj+1},\ w^*_iv^*_{jj+1}, u^*(v^*_{12} + \cdots + v^*_{n1}), v^*_{ii+1}w^*_j$ if $j \ne i, i+1$, and $v^*_{ii+1}(w^*_i + w^*_{i+1})$. The elements in the graded dual follow. $\square$

Now let us determine the trace on the graded pieces by seeing how each conjugacy class acts on the elements in the dual. As $A(\Gamma_{D_n})^! = A(\Gamma_{D_n})^!_{[0]} \oplus A(\Gamma_{D_n})^!_{[1]} \oplus A(\Gamma_{D_n})^!_{[2]} \oplus A(\Gamma_{D_n})^!_{[3]}$, there are only three degrees in the dual; so, we can calculate each independently.
Case 1: n=2m is even

The traces on the graded pieces are:

| | 1 | r | ... | $r^j$ | ... | $r^m$ | s | rs |
|---|---|---|---|---|---|---|---|---|
| $Tr_{\sigma,1}$ | 2n+1 | 1 | ... | 1 | ... | 1 | 3 | 3 |
| $Tr_{\sigma,2}$ | 2n-1 | -1 | ... | -1 | ... | -1 | -1 | -1 |
| $Tr_{\sigma,3}$ | 1 | 1 | ... | 1 | ... | 1 | -1 | -1 |

Now that we have the graded traces we can find the multiplicities of the representations by solving the system of equations: $\sum_\phi m_{\phi,i} * \chi_{\sigma\phi}(x) = Tr_{\sigma,i}(x)$, $x \in D_n$. They are:

| | $\chi_{triv}$ | $\chi_{1-1}$ | $\chi_{-11}$ | $\chi_{-1-1}$ | $\chi_k$ |
|---|---|---|---|---|---|
| $m_{\phi,1}$ | 3 | 0 | 1 | 1 | 2 |
| $m_{\phi,2}$ | 0 | 1 | 1 | 1 | 2 |
| $m_{\phi,3}$ | 0 | -1 | 0 | 0 | 0 |

Case 2: n=2m+1 is odd

The traces on the graded pieces are:

| | 1 | r | ... | $r^j$ | ... | $r^m$ | s |
|---|---|---|---|---|---|---|---|
| $Tr_{\sigma,1}$ | 2n+1 | 1 | ... | 1 | ... | 1 | 3 |
| $Tr_{\sigma,2}$ | 2n-1 | -1 | ... | -1 | ... | -1 | -1 |
| $Tr_{\sigma,3}$ | 1 | 1 | ... | 1 | ... | 1 | -1 |

The multiplicities are given below:

| | $\chi_{triv}$ | $\chi_{1-1}$ | $\chi_k$ |
|---|---|---|---|
| $m_{\phi,1}$ | 3 | 0 | 2 |
| $m_{\phi,2}$ | 0 | 1 | 2 |
| $m_{\phi,3}$ | 0 | -1 | 0 |

Notice that the graded traces and multiplicities are the same in both the even and odd cases.

These values give graded trace functions (in both even and odd cases) of:
$Tr_{(1)}(A(\Gamma_{D_n})^!, t) = 1 + (2n+1)t + (2n-1)t^2 + t^3$
$Tr_{r^i}(A(\Gamma_{D_n})^!, t) = 1 + t - t^2 + t^3$
$Tr_s(A(\Gamma_{D_n})^!, t) = Tr_{rs}(A(\Gamma_{D_n})^!, t) = 1 + 3t - t^2 - t^3$

## 8.2. **Representations of $S_n$ acting on $Q_n^!$**

[[GGRSW],§6] determines a basis for $Q_n^!$ as follows:
Let $A \subseteq \{1, ..., n\}$, $B$ be the sequence $(b_1, ..., b_k)$, and $B' = \{b_1, ..., b_k\}$. Define $S(A : B) = s(A)s(A\backslash b_1)\cdots s(A\backslash b_1\backslash ...\backslash b_k)$ where $s(A)$ is the image in $Q_n^!$ of the generator dual to $e(A, 1) \in Q_n$. Then $\mathcal{S} = \{S(A : B)|\text{min}A \notin B \text{ and } b_1 > \cdots > b_k\} \cup \{\emptyset\}$ is a basis for $Q_n^!$. The relations in the associated graded dual are:
1) $s(A)\sum_{a\in A} s(A\backslash a) = 0, |A| \geq 2$
2) $s(A)s(A\backslash i)s(A\backslash i\backslash j) = -s(A)s(A\backslash j)s(A\backslash i\backslash j)$
3) $s(A)s(B) = 0$ if $B \nsubseteq A$ or $|B| \neq |A| - 1$.

As opposed to the case of $Q_n$, $\sigma$ does not permute the basis elements of $Q_n^!$. Thus, it is not enough to count fixed basis elements to determine the trace. For each $S(A : B) \in \mathcal{S}$, we must write $\sigma S(A : B)$ as a linear combination of elements $\mathcal{S}$. Write this as $\sigma S(A : B) = S(\sigma A : \sigma B) = \sum_{S(C:D)\in\mathcal{S}} a_{\sigma A\sigma BCD} S(C : D)$. Then $\text{Tr}_\sigma = \sum_{S(A:B)\in\mathcal{S}} a_{\sigma A\sigma BAB}$. We are going to get three possible values for a basis element's contribution to the trace: $-1, 0,$ or $1$.

If $B$ is $\sigma$-invariant, then let $l_B(\sigma)$ be the number of pairs $i$, $j$ with $i < j$ and $\sigma b_i < \sigma b_j$. This is the length of $\sigma$ restricted to $B$. If there exists $c \in B$ such that $\sigma(c) = minA$, then define $\sigma' := (c\, minA)\sigma$.

**Proposition 8.2.**

$$a_{\sigma A\sigma BAB} = \begin{cases} (-1)^{l_B(\sigma)} & \textit{if } \sigma A = A, minA \notin \sigma B', \sigma B' = B' \\ (-1)^{l_B(\sigma')+1} & \textit{if } \sigma A = A, minA \in \sigma B' \textit{ and for some } b \in B', \sigma(B'\backslash b) = B'\backslash minA \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* If $\sigma \in S_n$ and $\sigma B' = B'$, then by relation (2) above we have that

$$S(A : \sigma B) = (-1)^{l_B(\sigma)} S(A : B). \tag{7}$$

If min $A \in B'$, then by relation (1)

$$S(A : B) = -\sum_{c \in A \setminus B'} S(A : (b_1, ..., b_{k-1}, c)). \tag{8}$$

Let us break this down into parts.
(*) $a_{ABCD} = 0$ if $C \neq A$. This is true because no relation changes the first factor, $s(C)$, of $S(C : D)$.
(**) $a_{ABCD} = 0$ unless $B' = D'$ or $B' = (B' \cap D') \cup \{minA\}$. Only relation (1) can change which elements are removed, and that relation can only change one element.

Recall $\sigma S(A : B) = S(\sigma A : \sigma B) = \sum_{S(C:D) \in \mathcal{S}} a_{\sigma A \sigma BCD} S(C : D)$. We need to know the value of $a_{\sigma A \sigma BAB}$. By (*) and (**), this is 0 unless $\sigma A = A$ and $\sigma B' = B'$ or $\sigma B' = (B' \cap \sigma B') \cup \{minA\}$. If $\sigma A = A$ and $\sigma B' = B'$, then, by Equation 7, $a_{\sigma A \sigma BAB} = (-1)^{l_B(\sigma)}$. If $\sigma A = A$ and $\sigma B' = (B' \cap \sigma B') \cup \{minA\}$, write $\sigma(b_j) = minA$. Then $(b_j\, minA)\sigma B' = \sigma' B' = B'$. Thus, by Equation 8, $\sigma S(A : B) = -S(A : (\sigma(b_1), ..., \hat{b_j}, ..., \sigma(b_k), b_j))$ + other terms. And, again by Equation 7, $\sigma S(A : B) = (-1)^{l_B(\sigma')}(-S(A : B))$. Hence, $a_{\sigma A \sigma BAB} = (-1)^{l_B(\sigma)+1}$. □

Now that we know what each basis element contributes to the trace, we want to find $Tr_\sigma(Q^!_n, t)$.

Let us introduce some notation. For $\sigma \in S_n$, write $\sigma = \sigma_1 \cdots \sigma_m$, a product of disjoint cycles. Denote the orbits of $\sigma$ by $\{\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_m\}$ and put an ordering on the orbits given by $\mathcal{O}_i < \mathcal{O}_j$ if the minimal element of $\mathcal{O}_i$ is less than that of $\mathcal{O}_j$. Say $\mathcal{O}_1 < \mathcal{O}_2 < \cdots < \mathcal{O}_m$. Let $i_j$ be the size of $\mathcal{O}_j$ (equal to the length of $\sigma_j$).

**Theorem 8.3.**

$$Tr_\sigma(Q^!_n, t) = \frac{1 + t\prod_{k=1}^{m}(2 - (-t)^{i_k})}{1 + t}$$

*Proof.* In order to prove the formula, we must take the sum over all $S(A : B) \in \mathcal{S}$ of $a_{\sigma A \sigma BAB}$, each of their contribution to the trace. We will do this in cases based on the value of the basis element's contribution to the trace.

Case 1: $\sigma A = A$, min$A \notin \sigma B'$, $\sigma B' = B'$

Consider $B = \mathcal{O}_{r_1} \cup ... \cup \mathcal{O}_{r_l}$, where $r_1 < \cdots < r_l$. Because $B' \subseteq A$ and min$A \notin \sigma B'$, $A$ must contain all of $\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}$ and at least one $\mathcal{O}_{r_0}$ such that $r_0 < r_1$ (so, $r_1 \neq 1$). Thus we must choose a nonempty subset of $\{\mathcal{O}_1, ..., \mathcal{O}_{r_1-1}\}$ and a subset $A'$ of $\{\mathcal{O}_{r_1+1}, ..., \mathcal{O}_m\}$ such that $A' \cap \{\mathcal{O}_{r_1+1}, ..., \mathcal{O}_m\} = \emptyset$. This gives $2^{m-r_1-(l-1)}(2^{r_1-1} - 1) = 2^{m-l} - 2^{m-r_1-l+1}$ choices for A. Because $l_B(\sigma_{r_i}) = i_{r_i} - 1$, $l_B(\sigma) = \sum_{1 \leq i \leq l} l_B(\sigma_{r_i}) = i_{r_1} + \cdots + i_{r_l} - l$. Also, the degree of $S(A : B)$ is $i_{r_1} + \cdots + i_{r_l} + 1$. Thus the contribution toward $Tr_\sigma$ for all such $S(A : B)$ given $B$ is $d(B) = (-1)^{i_{r_1}+\cdots+i_{r_l}-l}[2^{m-l} - 2^{m-r_1-l+1}]t^{i_{r_1}+\cdots+i_{r_l}+1}$. Summing over all $B$ in this case, we obtain

$$\sum_B d(B) = \sum_{\substack{2 \leq r_1 < \cdots < r_l \leq m \\ 1 \leq l \leq m-1}} (-1)^{i_{r_1}+\cdots+i_{r_l}-l}[2^{m-l} - 2^{m-r_1-l+1}]t^{i_{r_1}+\cdots+i_{r_l}+1}.$$

We will denote this by $c_1$ for ease of referencing later.

Case 2: $\sigma A = A$, $\min A \in \sigma B'$ and for some $b \in B'$, $\sigma(B' \backslash b) = B' \backslash \min A$

Fix $B$, say $B' \subset \mathcal{O}_{r_1} \cup ... \cup \mathcal{O}_{r_l}$. $B'$ must contain all elements in $\{\mathcal{O}_{r_2}, ..., \mathcal{O}_{r_l}\}$ since $B'$ and $\sigma B'$ can only differ by one element; and, that must occur in $\mathcal{O}_{r_1}$ because $\min A$ must be in $\mathcal{O}_{r_1}$ and cannot be in $B'$. Say $\sigma_{r_1} = (c_{i_{r_1}-1} \cdots c_1\, minA)$. Then $B$ must also contain consecutive elements $\{c_1, ..., c_j\}$, $1 \leq j \leq i_{r_1} - 1$, in $\mathcal{O}_{r_1}$. If this were not the case, $B'$ and $\sigma B'$ would differ by more than one element ($c_j \notin \sigma B$).

Consider $\sigma' = (c_j\, minA)(minA\, c_{i_{r_1}-1} \cdots c_1)\sigma_{r_2} \cdots \sigma_{r_1} =$

$(c_j\, minA)(minA\, c_1) \cdots (minA\, c_{i_{r_1}-1})\sigma_{r_2} \cdots \sigma_{r_1}$. Then $l_B(\sigma') = \sum_{k=2}^{l}(i_{r_k} - 1) + j + 1$. Thus, by Proposition 8.2, the trace of $\sigma$ acting on $S(A : B)$ is $(-1)^{j+i_{r_2}+\cdots+i_{r_l}-(l-1)+2} = (-1)^{j+i_{r_2}+\cdots+i_{r_l}-(l-1)}$.

Given $B$, $A$ must contain $\{\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}\}$ and may contain other orbits greater than $\mathcal{O}_{r_1}$. Thus, there are $2^{m-r_1-(l-1)}$ choices for $A$. Now there are $i_{r_1} - 1$ subsets $B \subset \mathcal{O}_{r_1} \cup ... \cup \mathcal{O}_{r_l}$. Putting this all together, given $\{\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}\}$, $S(A : B)$ contributes a total of

$2^{m-r_1-(l-1)} \sum_{j=1}^{i_{r_1}-1} (-1)^{j+i_{r_2}+\cdots+i_{r_l}-(l-1)} t^{j+i_{r_2}+\cdots+i_{r_l}+1}$ towards the graded trace function.

We will need to sum this over all $\{\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}\}$ and multiply by $1 + t$. This gives us

$$\sum_{\substack{1 \leq r_1 < \cdots < r_l \leq m \\ 1 \leq l \leq m}} 2^{m-r_1-l+1}(-1)^{i_{r_2}+\cdots+i_{r_l}-l}\, t^{i_{r_2}+\cdots+i_{r_l}+2}$$
$$+ \sum_{\substack{1 \leq r_1 < \cdots < r_l \leq m \\ 1 \leq l \leq m}} 2^{m-r_1-l+1}(-1)^{i_{r_1}+\cdots+i_{r_l}-l}\, t^{i_{r_1}+\cdots+i_{r_l}+1}$$

(notice that the sum over $j$ is telescoping.) Let us label the first sum by $c_2$ and the second by $c_3$ for ease of referencing later.

Case 3: $B = \emptyset$.

Because $\sigma A = A$, $a_{\sigma A \sigma B A B} = 1$. Thus we have a contribution of $1 + (2^m - 1)t$ towards the graded trace. Multiplying by $1 + t$ gives us $1 + 2^m t + (2^m - 1)t^2$.

If we sum over all possibilities for the traces and multiply by $1 + t$, we have that $Tr_\sigma(Q_n^!, t) = 1 + 2^m t + (2^m - 1)t^2 + c_1 + c_1 t + c_2 + c_3$.

Consider the following pieces of the expression.

First sum $(2^m - 1)t^2$ and the $l = 1$ terms of $c_2$.

$$(2^m - 1)t^2 + c_2|_{l=1} = (2^m - 1)t^2 + \sum_{1 \leq r_1 \leq m} 2^{m-r_1}(-1)^{-1}t^2$$
$$= t^2[(2^m - 1) - \sum_{r_1=1}^{m} 2^{m-r_1}] = t^2[(2^m - 1) - (2^{m-1+1} - 1)] = 0.$$

Next sum the remaining terms of $c_2$ (with $l > 1$) and $tc_1$ where we do a change of variables setting $r_1$ to $r_2$.

$$
\begin{aligned}
&c_1 t|_{r_1 \mapsto r_2} + c_2|_{l>1} = \\
&(2^{m-l+1} - 2^{m-r_2-l+1+1})(-1)^{i_{r_2}+\cdots+i_{r_l}-(l-1)} t^{i_{r_2}+\cdots+i_{r_l}+2} \\
&\quad + \sum_{r_1=1}^{r_2-1} 2^{m-r_1-l+1}(-1)^{i_{r_2}+\cdots+i_{r_l}-l} t^{i_{r_2}+\cdots+i_{r_l}+2} \\
&\quad = (-1)^{i_{r_2}+\cdots+i_{r_l}-l+1}\, t^{i_{r_2}+\cdots+i_{r_l}+2}[2^{m-l+1} - 2^{m-r_2-l+2} - \sum_{r_1=0}^{r_2-2} 2^{m-r_1-l}] \\
&\quad = (-1)^{i_{r_2}+\cdots+i_{r_l}-l+1}\, t^{i_{r_2}+\cdots+i_{r_l}+2}[2^{m-l+1} - 2^{m-r_2-l+2} - [\sum_{r_1=0}^{m-l} 2^{r_1} - \sum_{r_1=0}^{m-r_2-l+1} 2^{r_1}]] \\
&\quad = (-1)^{i_{r_2}+\cdots+i_{r_l}-l+1}\, t^{i_{r_2}+\cdots+i_{r_l}+2}[2^{m-l+1} - 2^{m-r_2-l+2} - (2^{m-l+1}-1) + (2^{m-(r_2-1)-l+1}-1)] \\
&\quad = 0.
\end{aligned}
$$

Finally sum $c_1$ and the terms of $c_3$ with $r_1 \neq 1$.

$$
\begin{aligned}
c_1 + c_3|_{r_1 \neq 1} &= \sum_{\substack{1 \le r_1 < \cdots < r_l \le m \\ 1 \le l \le m-1}} (-1)^{i_{r_1}+\cdots+i_{r_l}-l} t^{i_{r_1}+\cdots+i_{r_l}+1}[2^{m-l} - 2^{m-r_1-l+1} - 2^{m-r_1-l+1}] \\
&= \sum_{\substack{1 \le r_1 < \cdots < r_l \le m \\ 1 \le l \le m-1}} (-1)^{i_{r_1}+\cdots+i_{r_l}-l}\, t^{i_{r_1}+\cdots+i_{r_l}+1} 2^{m-l}.
\end{aligned}
$$

The terms of $c_3$ with $r_1 = 1$ are: $\displaystyle\sum_{1 \le l \le m} (-1)^{i_1+\cdots+i_{r_l}-l}\, t^{i_1+\cdots+i_{r_l}+1} 2^{m-l}$.

Putting it all together we obtain:

$$
1 + t[2^m + \sum_{\substack{1 \le r_1 < \cdots < r_l \le m \\ 1 \le l \le m}} (-1)^l 2^{m-l}(-t)^{i_{r_1}+\cdots+i_{r_l}}] = 1 + t\prod_{k=1}^{m}(2 - (-t)^{i_k})
$$

Therefore,

$$
Tr_\sigma(Q_n^!, t) = \frac{1 + t\prod_{k=1}^{m}(2 - (-t)^{i_k})}{1+t}
$$

as desired. $\square$

**Example 8.1.** *Here are the graded trace functions for $Q_4^!$:*

$$
Tr_{(1)}(Q_4^!, t) = \frac{1 + t(2+t)^4}{1+t} = 1 + 15t + 17t^2 + 7t^3 + t^4
$$

$$
Tr_{(12)}(Q_4^!, t) = \frac{1 + t(2-t^2)(2+t)^2}{1+t} = 1 + 7t + t^2 - 3t^3 - t^4
$$

$$
Tr_{(123)}(Q_4^!, t) = \frac{1 + t(2+t^3)(2+t)}{1+t} = 1 + 3t - t^2 + t^3 + t^4
$$

$$
Tr_{(12)(34)}(Q_4^!, t) = \frac{1 + t(2-t^2)^2}{1+t} = 1 + 3t - 3t^2 - t^3 + t^4
$$

$$
Tr_{(1234)}(Q_4^!, t) = \frac{1 + t(2-t^4)}{1+t} = 1 + t - t^2 + t^3 - t^4
$$

Now, to get the representations we do the same as in the case of the algebra. We have that $\vec{m}(t) = (S^T)^{-1}\vec{Tr}(t)$ and Proposition 7.2 are still true if you replace $Tr_\sigma(Q_n, t)$ with $Tr_\sigma(Q_n^!, t)$.

**Example 8.2.** *Irreducible Representations of $S_4$ acting on $Q_4^!$:*

*There are only four degrees in the dual, so we can give all of the multiplicities:*

| | $\chi_{triv}$ | $\chi_{sgn}$ | $\chi_3$ | $\chi_{reg}$ | $\chi_{sgn\otimes reg}$ |
|---|---|---|---|---|---|
| $m_{\phi,1}$ | 4 | 0 | 1 | 3 | 0 |
| $m_{\phi,2}$ | 0 | 0 | 1 | 3 | 2 |
| $m_{\phi,3}$ | 0 | 1 | 0 | 0 | 2 |
| $m_{\phi,4}$ | 0 | 1 | 0 | 0 | 0 |

Notice that all of the representations are realized in at least one grading, but not in every. Also, each representation occurs with a much smaller multiplicity than in the algebra.

# 9. Koszulity property

An interesting property of quadratic algebras is Koszulity. One of many equivalent definitions of Koszulity is a lattice definition [F].

**Definition** (Koszul Algebra)**.** [B] *Let $A = (V, R)$ be a quadratic algebra where $V$ is the span of the generators and $R$ the span of the generating relations in $V \otimes V$. Then $A$ is Koszul if the collection of subspaces $\{V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2}, 0 \le i \le n-2\}$ generates a distributive lattice in $V^{\otimes n}$ for any $n$.*

One property of Koszul algebras is that the Hilbert series of the algebra and its dual are related by $H(A, t) * H(A^!, -t) = 1$. This property, however, is not equivalent to Koszulity. One can easily check that the analogous property holds for the graded trace functions that we found for $A(\Gamma)$ and its dual $A(\Gamma)^!$ in our two algebras. Namely, $Tr_\sigma(A(\Gamma), t) * Tr_\sigma(A(\Gamma)^!, -t)$=1 where $\sigma$ is an element in the automorphism group of the algebra.

# References


[B] Backelin, J. *A distributiveness property of augmented algebras and some related homological results*. Ph.D. thesis, Stockholm University (1982).

[BVW] Berger, R., M. Dubois-Violette, and M. Wambst. *Homogeneous Algebras*. arxiv:math.QA/0203035 (2002).

[Berg] Bergman, G. *The diamond lemma for ring theory*. Advances in Mathematics **29**(2): 178–218 (1978).

[D] Duffy, C. *Graded traces and irreducible representations of* $Aut(A(\Gamma))$ *acting on* $grA(\Gamma)$ *and* $grA(\Gamma)^!$. Ph.D. thesis, Rutgers University (2008).

[F] Froberg, R. *Koszul Algebras*. Lecture Notes in Pure and Appl. Math. **205**: 337–350 (1999).

[GGRSW] Gelfand, I., S. Gelfand, V. Retakh, S. Serconek, and R. L. Wilson. *Hilbert series of quadratic algebras associated with pseudo-roots of noncommutative polynomials*. Journal of Algebra **254**: 279–299 (2002).

[GGRW] Gelfand, I., S. Gelfand, V. Retakh, and R. L. Wilson. *Factorizations of Polynomials over Noncommutative Algebras and Sufficient Sets of Edges in Directed Graphs* **74**(2): 153 (2005).

[GRSW] Gelfand, I., V. Retakh, S. Serconek, and R. L. Wilson. *On a class of algebras associated to directed graphs*. Selecta Math. **11**: 281–295 (2005).

[Pr] Priddy, S. *Koszul Resolutions*. Trans. Amer. Math. Soc. **152**: 39–60 (1970).

[RSW3] Retakh, V., S. Serconek, and R. L. Wilson. *Construction of some algebras associated to directed graphs and related to factorizations of noncommutative polynomials*. Contemporary Math. **442** (2006).

[RSW2] Retakh, V., S. Serconek, and R. L. Wilson. *On a class of Koszul algebras associated to directed graphs*. J. of Algebra **304**: 1114–1129 (2006).

[RSW] Retakh, V., S. Serconek, and R. L. Wilson. *Hilbert series of algebras associated to directed graphs*. J. of Algebra **312**: 142–151 (2007).

[RW] Retakh, V. and R. L. Wilson. *Algebras associated to directed acyclic graphs*. arxiv:math.CO/0707.3607 (2007).

[Wilf] Wilf, H. S. *generatingfunctionology*. Academic Press, Inc. (1990).



Rutgers, the State University of New Jersey, Department of Mathematics - Hill Center, 110 Frelinghuysen Rd., Piscataway, NJ 08854

*E-mail address*: duffyc@math.rutgers.edu